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ALMOST PERIODIC AND ALMOST AUTOMORPHIC DYNAMICS FOR SCALAR CONVEX DIFFERENTIAL EQUATIONS*

BY

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ABSTRACT

We study the set of bounded trajectories for the flow defined by a class of scalar convex differential equations depending on a parameter. It is found that there exists precisely one value of the parameter for which almost automorphic but not almost periodic dynamics may appear. Even for this parameter value, the occurrence of almost periodic dynamics is shown to be residual in some cases. The dependence of this parameter on the functions defining the differential equations is also studied.

1. Introduction

A well-known result asserts that a scalar periodic differential equation which admits a bounded solution also admits a periodic solution (see Massera [17]). The result is no longer true for an almost periodic differential equation. However, if this equation admits a bounded solution, then it also admits a recurrent

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solution which generates an almost automorphic minimal set. Almost automorphic functions were first introduced by Bochner [4] in 1956. These functions generalize almost periodic functions; they admit a well-defined Fourier series although not necessarily unique, and the respective Bochner–Féjer sums only converge pointwise in general (see Veech [30]). The closure of an almost automorphic orbit in a dynamical system is minimal, but its measurable structure can exhibit high complexity in some cases (see Furstenberg and Weiss [9]).

Examples of a quadratic almost periodic differential equation with almost automorphic but not almost periodic solutions can be deduced from the non-hyperbolic bidimensional linear systems obtained by Millionščikov [19, 20] and Vinograd [32]. It is easy to check that the projective flow induced by these examples can be described by a strictly concave differential equation. A simple change of sign in the independent variable provides an alternative strictly convex representation. In this same line, the papers of Zhikov and Levitan [33] and Johnson [11] contain examples of a scalar linear almost periodic differential equation showing similar phenomena. Ortega and Tarallo [23] describe a qualitative property, which is satisfied by the above examples, and it provides almost automorphic solutions of almost periodic scalar linear equations.

This paper deals with the presence of almost periodic and almost automorphic dynamics in the local flows defined by a one parameter family of strictly convex differential equations

$$x' = g(\omega \cdot t, x) + \alpha, \quad \omega \in \Omega, \tag{1.1}$$

where $\omega \cdot t = \sigma(t, \omega)$ represents a continuous flow on a compact metric space Ω, and the map $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and coercive, i.e., $\lim_{x\to\pm\infty} g(\omega, x) = \infty$ for every $\omega \in \Omega$. If the base (Ω, σ) is almost periodic, we apply the dynamical description of the bounded trajectories set obtained by Alonso and Obaya [1], to show that the existence of almost automorphic but not almost periodic solutions is only possible for a unique value of the parameter α^*. Besides, when (1.1) takes the form

$$x' = g(x) + p(\omega \cdot t) + \alpha, \quad \omega \in \Omega, \tag{1.2}$$

we prove that for this value of the parameter $\alpha^*(p)$, generically in an appropriate set of functions p, there exists a unique bounded solution which is almost periodic. Previous results for convex (resp. concave) coercive scalar differential equations were obtained by Mawhin [18] (resp. Tineo [29]), where the existence and number of periodic solutions (resp. bounded separate solutions) depending on the parameter α were studied. Novo and Obaya [22] adapt the techniques of Alonso and Obaya [1] to describe hyperbolic almost periodic minimal sets in infinite dimensional convex monotone dynamical systems.

Finally, we mention other points of interest concerning almost automorphic dynamics. Floquet theory for almost periodic linear systems must be formulated in terms of almost automorphic linear transformations, as shown in Johnson [10]. Shen and Yi [27] illustrate the relevance of the almost automorphic dynamics in the study of infinite dimensional almost periodic differential equations. Its influence in the case of monotone structures is also remarkable, as shown by Johnson et al. [13, 14], Shen and Yi [28] and the references therein. The complexity of almost automorphic dynamics has been studied for symbolic flows in Furstenberg [8] and Markley and Paul [16]. Recent results in high dimensional symbolic flows with application to the construction of almost automorphic chaotic signals can be found in Berger et al. [3].

This paper is arranged as follows. Section 2 reviews some basic notions and well-known results in ergodic theory and topological dynamics. It also contains a sketch of the proof of a perturbation theorem often used through the paper.

In Section 3, under some assumptions of strict convexity, we prove that if Ω is almost periodic, there is a unique value of the parameter α^* for which the corresponding equation (1.1) may admit an almost automorphic but not almost periodic solution. We also characterize the ergodic and topological structure of the set of bounded trajectories for this value of the parameter. In addition, similar results are obtained when we only require the convexity properties for the function g over a compact invariant set of bounded trajectories with nonempty interior.

Section 4 studies the dependence on the function p of the value of the parameter α^* for equation (1.2), obtaining a continuity result for the uniform convergence and a kind of semicontinuity result for the weak topology $\sigma(C(\Omega), M(\Omega))$. A characterization of the convergence of the Lyapunov exponents in terms of the convergence in measure of the hyperbolic solutions is also obtained.

After recalling some notions on almost periodic functions, Section 5 is devoted to the study of the case in which Ω is the hull of a limit periodic function. We show that equation (1.2) for $\alpha^*(p)$ admits an almost periodic solution for p in a residual subset of $C(\Omega)$. Finally, Section 6 deals with the quasi-periodic case, where also the genericity of almost periodic dynamics is shown in the product space of frequencies and continuous functions $[0,1]^k \times C(\mathbb{T}^k)$.

2. Basic results

This section is devoted to recall some definitions and results more or less standard in ergodic theory and topological dynamics, as well as to state some basic results used through the paper.

Let Ω be a compact metric space. A real **continuous flow** (Ω, σ) is defined by a continuous mapping $\sigma\colon \mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \sigma(t, \omega)$ satisfying

(i) $\sigma_0 = \mathrm{Id}$,

(ii) $\sigma_{t+s} = \sigma_t \circ \sigma_s$ for each s, $t \in \mathbb{R}$,

where $\sigma_t(\omega) = \sigma(t, \omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$. The set $\{\sigma_t(\omega) \mid t \in \mathbb{R}\}$ is called the **orbit** or the **trajectory** of the point ω.

We say that a subset $\Omega_1 \subset \Omega$ is **σ-invariant** if $\sigma_t(\Omega_1) = \Omega_1$ for every $t \in \mathbb{R}$. A mapping $f\colon \Omega \to \mathbb{R}$ is **σ-invariant** if it is constant along the trajectories, i.e., $f(\sigma_t(\omega)) = f(\omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$. A subset $\Omega_1 \subset \Omega$ is called **minimal** if it is compact, σ-invariant and it has no other nonempty compact σ-invariant subset but itself. Every compact and σ-invariant set contains a minimal subset; in particular, it is easy to prove that a compact σ-invariant subset is minimal if and only if every trajectory is dense. We say that the continuous flow (Ω, σ) is **recurrent** or **minimal** if Ω is minimal.

Let d be a metric on Ω. The flow (Ω, σ) is said to be **almost periodic** when for every $\varepsilon > 0$ there is a $\delta > 0$ such that, if ω_1, $\omega_2 \in \Omega$ with $\mathsf{d}(\omega_1, \omega_2) < \delta$, then $\mathsf{d}(\sigma_t(\omega_1), \sigma_t(\omega_2)) < \varepsilon$ for every $t \in \mathbb{R}$. For the basic properties on almost periodic flows we refer the reader to Ellis [6] and Sacker and Sell [25].

A **flow homomorphism** from another continuous flow (Y, Ψ) to (Ω, σ) is a continuous mapping $f\colon Y \to \Omega$ such that $f(\Psi_t(y)) = \sigma_t(f(y))$ for every $y \in Y$ and $t \in \mathbb{R}$. Let $\pi\colon (Y, \Psi) \to (\Omega, \sigma)$ be a surjective flow homomorphism and suppose (Y, Ψ) is minimal. We say that (Y, Ψ) is a **copy** of (Ω, σ) if $\mathrm{card}(\pi^{-1}(\omega)) = 1$ for each $\omega \in \Omega$. (Y, Ψ) is said to be an **almost automorphic extension** (a.a. extension) of (Ω, σ) if there is $\omega \in \Omega$ such that $\mathrm{card}(\pi^{-1}(\omega)) = 1$. A minimal flow (Y, Ψ) is **almost automorphic** if it is an almost automorphic extension of an almost periodic minimal flow (Ω, σ) (see Veech [31]).

A Borel measure on Ω will be a finite regular measure defined on the Borel sets. Let μ be a normalized Borel measure on Ω; μ is **σ-invariant** (or **invariant under** σ) if $\mu(\sigma_t(\Omega_1)) = \mu(\Omega_1)$ for every Borel subset $\Omega_1 \subset \Omega$ and every $t \in \mathbb{R}$. It is **σ-ergodic** (or **ergodic under** σ) if, in addition, $\mu(\Omega_1) = 0$ or $\mu(\Omega_1) = 1$ for every σ-invariant subset $\Omega_1 \subset \Omega$.

We denote by $\mathcal{M}_{\mathrm{inv}}(\Omega, \sigma)$ the set of positive and normalized σ-invariant measures on Ω. The Krylov–Bogoliubov theorem (see Nemytskii and Stepanoff [21]) asserts that $\mathcal{M}_{\rm inv}(\Omega,\sigma)$ is nonempty when Ω is a compact metric space. The extremal points of the convex and weakly compact set $\mathcal{M}_{\rm inv}(\Omega,\sigma)$ are the σ-ergodic measures, from which it is deduced that also the set of σ-ergodic measures is nonempty. The decomposition of the flow (Ω,σ) into ergodic components and the construction and representation theorems of σ-invariant measures from σ-ergodic measures are well known (see Phelps [24] and Mañé [15]).

We say that (Ω,σ) is **uniquely ergodic** (u.e.) if it has a unique normalized invariant measure which is then necessarily ergodic. If (Ω,σ) is u.e. it is not necessarily minimal; however, if (Ω,σ) is u.e. and $\mu(U)>0$ for every non-empty open set U, then (Ω,σ) is minimal. An almost periodic and minimal flow (Ω,σ) is always u.e.

Next we consider a family of scalar ordinary differential equations

$$x' = h(\omega\cdot t, x), \quad \omega\in\Omega, \tag{2.1}$$

where $h\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is continuous, differentiable with respect to x and $\partial h/\partial x$ is continuous. They induce a local skew-product flow on $\Omega\times\mathbb{R}$

$$\begin{array}{rlcl} \tau & : U\subset\mathbb{R}\times\Omega\times\mathbb{R} & \longrightarrow & \Omega\times\mathbb{R} \\ & (t,\omega,x_0) & \mapsto & \tau(t,\omega,x_0)=(\omega\cdot t, x(t,\omega,x_0)), \end{array}$$

where $x(t,\omega,x_0)$ is the solution of (2.1) evaluated along the trajectory of ω with initial value x_0, and t belongs to its maximal interval of definition.

A forward orbit $\{\tau(t,\omega_0,x_0)|t\ge 0\}$ is **uniformly stable** if for every $\varepsilon>0$ there is a $\delta=\delta(\varepsilon)>0$, called the **modulus of uniform stability**, such that if $s\ge 0$ and $|x(s,\omega_0,x_0)-x(s,\omega_0,x_1)|\le\delta(\varepsilon)$, then

$$|x(t+s,\omega_0,x_0)-x(t+s,\omega_0,x_1)|\le\varepsilon \quad \text{for each } t\ge 0.$$

It is said to be **uniformly asymptotically stable** if it is uniformly stable and there is a $\delta_0>0$ with the following property: for each $\varepsilon>0$ there is a $t_0(\varepsilon)>0$ such that if $s\ge 0$ and $|x(s,\omega_0,x_0)-x(s,\omega_0,x_1)|\le\delta_0$ then

$$|x(t+s,\omega_0,x_0)-x(t+s,\omega_0,x_1)|\le\varepsilon \quad \text{for each } t\ge t_0(\varepsilon).$$

Let M be a minimal subset of the above local flow. The natural projection π: $(M,\tau)\to(\Omega,\sigma)$, $(\omega,x)\mapsto\omega$ defines a flow homomorphism. When the base (Ω,σ) is minimal, it is well known that (M,τ) is an almost automorphic extension of (Ω,σ) (see Shen and Yi [28] for a more general version of this result, valid for strongly monotone semiflows).

We say that $x_0 \in C(\Omega)$ is a hyperbolic solution of (2.1) if it is a solution along the trajectories, that is, $x_0(\omega\cdot t)$ satisfies equation (2.1) for each $\omega \in \Omega$, and the linearized family

$$z' = \frac{\partial h}{\partial x}(\omega\cdot t, x_0(\omega\cdot t))z, \quad \omega \in \Omega \tag{2.2}$$

admits an exponential dichotomy over Ω. This means that there are positive constants $C > 0$, $\beta > 0$ and a continuous family of projections P_ω such that if $\Phi_\omega(t)$ denotes the fundamental solution of (2.2),

$$\begin{aligned} |\Phi_\omega(t)P_\omega\Phi_\omega^{-1}(s)| &\le Ce^{-\beta(t-s)} \qquad \text{if } t \ge s,\\ |\Phi_\omega(t)(I-P_\omega)\Phi_\omega^{-1}(s)| &\le Ce^{\beta(t-s)} \quad\ \text{if } t \le s. \end{aligned}$$

In the scalar case there are two possibilities: $P_\omega = I$ or $P_\omega = 0$ for all $\omega \in \Omega$. Consequently,(2.2) admits an exponential dichotomy over Ω if and only if

$$\exp\int_s^t \frac{\partial h}{\partial x}(\omega\cdot u, x_0(\omega\cdot u))du \le Ce^{-\beta(t-s)} \quad \text{for each } t \ge s, \omega \in \Omega$$

or

$$\exp\int_s^t \frac{\partial h}{\partial x}(\omega\cdot u, x_0(\omega\cdot u))du \le Ce^{\beta(t-s)} \quad \text{for each } t \le s, \omega \in \Omega.$$

When the base (Ω, σ) is minimal, we say that a minimal subset $M \subset \Omega\times\mathbb{R}$ is hyperbolic if $M = \{(\omega, x_0(\omega)) \mid \omega \in \Omega\}$ with $x_0 \in C(\Omega)$ a hyperbolic solution. Thus, a hyperbolic minimal subset is always a **copy of the base**.

Let m be a fixed ergodic measure on Ω and B a compact invariant subset of $\Omega\times\mathbb{R}$. We consider the global flow on B defined by the above map τ. We say that γ is a **Lyapunov exponent with respect to** m if there exists an ergodic measure ν projecting onto m such that

$$\lim_{|t|\to\infty}\frac{1}{t}\int_0^t \frac{\partial h}{\partial x}(\omega\cdot s, x(s,\omega,x_0))ds = \gamma \quad \text{a.e. with respect to } \nu.$$

Birkhoff's ergodic theorem leads us to $\int_B \partial h/\partial x d\nu = \gamma$.

The following perturbation theorem asserts that hyperbolicity is maintained in an appropriate neighborhood, and it will be often used along the paper. Its proof is based on Theorem 8.1 of Fink [7] and Lemma 3.3 of Alonso et al. [2]. Since we will need to apply the result jointly to more than one family, we include a sketch of the proof to remark how the neighborhood depends mainly on the constants of the exponential dichotomy.

THEOREM 2.1: *We consider* (Ω, σ) *a continuous flow (not necessarily minimal) on a compact metric space. Let us assume that* $x_0 \in C(\Omega)$ *is a hyperbolic solution of*

$$x' = h(\omega \cdot t, x), \quad \omega \in \Omega,$$

where $h\colon \Omega \times \mathbb{R} \to \mathbb{R}$ *is continuous, differentiable with respect to* x *and* $\partial h/\partial x$ *is continuous. Then, for each* $\delta > 0$*, there is an* $\varepsilon(\delta) > 0$ *such that if* $p \in C(\Omega)$ *and* $\|p\|_\infty < \varepsilon(\delta)$*, the perturbed family*

$$x' = h(\omega \cdot t, x) + p(\omega \cdot t), \quad \omega \in \Omega, \tag{2.3}$$

admits a hyperbolic solution $x_p \in C(\Omega)$ *with* $\|x_0 - x_p\|_\infty < \delta$*.*

Proof: We give a sketch of the proof. Let $p \in C(\Omega)$ and consider the family of equations (2.3). The change of variables $z = x - x_0(\omega \cdot t)$ takes it into

$$z' = \frac{\partial h}{\partial x}(\omega \cdot t, x_0(\omega \cdot t))z + r(\omega \cdot t, z) + p(\omega \cdot t), \quad \omega \in \Omega, \tag{2.4}$$

where $r(\omega, z) = h(\omega, z + x_0(\omega)) - h(\omega, x_0(\omega)) - \partial h/\partial x(\omega, x_0(\omega))z$.

We define $L(z)$ for each real z by

$$\sup\Big\{\Big|\frac{\partial h}{\partial x}(\omega, z_1) - \frac{\partial h}{\partial x}(\omega, z_2)\Big| \mid \omega \in \Omega, |z_1|, |z_2| \le \|x_0\|_\infty + 1, |z_1 - z_2| < |z|\Big\}.$$

It is not difficult to see that $\lim_{z\to 0} L(z) = 0$. Besides, the following relations hold for every $\omega \in \Omega$: $|r(\omega, z)| \le L(z)|z|$ whenever $|z| \le 1$ and

$$|r(\omega, z_1) - r(\omega, z_2)| \le L(\max\{|z_1|, |z_2|\})|z_1 - z_2|,$$

provided that $|z_1|, |z_2| \le 1$.

Since x_0 is a hyperbolic solution of the original family, the linearized (2.2) admits an exponential dichotomy over Ω. Moreover, if $y \in C(\Omega)$ and $\|x_0 - y\|_\infty < \delta$,

$$\Big|\frac{\partial h}{\partial x}(\omega \cdot t, x_0(\omega \cdot t)) - \frac{\partial h}{\partial x}(\omega \cdot t, y(\omega \cdot t))\Big| \le L(\delta), \quad \text{for each } t \in \mathbb{R}, \omega \in \Omega.$$

Thus, since $\lim_{\delta\to 0} L(\delta) = 0$, we can assume that δ is sufficiently small, depending only on the constants of the exponential dichotomy, to apply the classical roughness theorem (see Coppel [5]), and conclude that the linear systems

$$z' = \frac{\partial h}{\partial x}(\omega \cdot t, y(\omega \cdot t))z, \quad \omega \in \Omega$$

also admit an exponential dichotomy over Ω.

For each $z_0 \in \overline{B}(0,\delta) \subset C(\Omega)$, we consider the linear equation with bounded nonhomogeneous term

$$z' = \frac{\partial h}{\partial x}(\omega\cdot t, x_0(\omega\cdot t))z + r(\omega\cdot t, z_0(\omega\cdot t)) + p(\omega\cdot t), \quad t\in\mathbb{R}, \tag{2.5}$$

which admits a unique bounded solution $T_{z_0,\omega}(t)$ because of the exponential dichotomy of (2.2) (see also [5]). Moreover, $\|T_{z_0,\omega}\|_\infty \le K(L(\delta)\delta + \|p\|_\infty)$, where the constant K only depends on the constants involved in the exponential dichotomy. Next, we define

$$T\colon \overline{B}(0,\delta) \longrightarrow C(\Omega), \quad z_0 \mapsto Tz_0,$$

where $Tz_0(\omega) = T_{z_0,\omega}(0)$. It is also easy to check that $Tz_0(\omega\cdot t) = T_{z_0,\omega}(t)$ for each $t\in\mathbb{R}$, and for each z_1, $z_2 \in \overline{B}(0,\delta)$

$$\|Tz_1\|_\infty \le K(L(\delta)\delta + \|p\|_\infty) \quad \text{and} \quad \|Tz_1 - Tz_2\| \le KL(\delta)\|z_1 - z_2\|_\infty.$$

Since $\lim_{\delta\to 0} L(\delta) = 0$, we can reduce, if necessary, the size of δ and choose an $\varepsilon(\delta)$ in order to get a contraction from $\overline{B}(0,\delta)$ into itself when $\|p\|_\infty < \varepsilon(\delta)$.

Finally, the Banach fixed point theorem provides a bounded solution $z_p(\omega\cdot t)$ of equation (2.4). Therefore, $x_p = z_p + x_0$ is a bounded solution of (2.3) satisfying $\|x_0 - x_p\|_\infty < \delta$, which, as explained before, implies that it is a hyperbolic solution. Notice that $\varepsilon(\delta)$ only depends on the constants of the exponential dichotomy and on the partial derivative $\partial h/\partial x$. ∎

3. Scalar convex differential equations

Let (Ω,σ) be a minimal flow defined on a compact metric space and represent $\omega\cdot t = \sigma(t,\omega)$ for each $t\in\mathbb{R}$ and $\omega\in\Omega$. We consider a continuous map satisfying the following properties.

Assumption 3.1: The continuous function $g\colon \Omega\times\mathbb{R}\to\mathbb{R}$ satisfies

(a.1) g is a convex map in the x component, that is,

$$g(\omega, \lambda x_1 + (1-\lambda)x_2) \le \lambda g(\omega, x_1) + (1-\lambda) g(\omega, x_2)$$

for every $\omega\in\Omega, 0\le\lambda\le 1$ and x_1, $x_2\in\mathbb{R}$, and strictly convex in the x component for some $\omega_0\in\Omega$, i.e.,

$$g(\omega_0, \lambda x_1 + (1-\lambda)x_2) < \lambda g(\omega_0, x_1) + (1-\lambda) g(\omega_0, x_2)$$

for every $0 < \lambda < 1$ and $x_1, x_2 \in \mathbb{R}$ with $x_1 \neq x_2$;

(a.2) g is differentiable with respect to x and $\partial g/\partial x\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous;

(a.3) g is coercive in x, i.e., $\lim_{x\to\pm\infty} g(\omega, x) = \infty$ for every $\omega \in \Omega$.

For each $\alpha \in \mathbb{R}$, the family of differential equations

$$x' = g(\omega \cdot t, x) + \alpha, \quad \omega \in \Omega \tag{3.1$_\alpha$}$$

induces a local skew-product flow on $\Omega \times \mathbb{R}$

$$\begin{array}{rlcl} \tau_\alpha & : U \subset \mathbb{R} \times \Omega \times \mathbb{R} & \longrightarrow & \Omega \times \mathbb{R} \\ & (t, \omega, x_0) & \mapsto & \tau_\alpha(t, \omega, x_0) = (\omega \cdot t, x(t, \omega, x_0, \alpha)), \end{array}$$

where $x(t, \omega, x_0, \alpha)$ is the solution of $(3.1)_\alpha$ evaluated along the trajectory of ω with initial value x_0, and t belongs to its maximal interval of definition (t_-, t_+) (notice that, although dropped from the notation, t_+ and t_- depend on ω, x_0 and α). It is well known that if $x(t, \omega, x_0, \alpha)$ remains bounded, then it is defined for every $t \in \mathbb{R}$. We consider the set of bounded solutions

$$B_\alpha = \{(\omega, x_0) \in \Omega \times \mathbb{R} \mid \sup_{t\in\mathbb{R}} |x(t, \omega, x_0, \alpha)| < \infty\},$$

and we denote $\pi\colon B_\alpha \mapsto \Omega$ as the natural projection. Let us assume that $B_\alpha \neq \emptyset$ and let $(\omega_0, x_0) \in B_\alpha$. Thus, $\mathrm{cls}\{(\omega_0 \cdot t, x(t, \omega_0, x_0, \alpha)) \mid t \in \mathbb{R}\} \subset B_\alpha$ and the minimal character of (Ω, σ) provides $\pi^{-1}(\omega) \cap B_\alpha \neq \emptyset$ for every $\omega \in \Omega$. Therefore, the map τ_α defines a global flow on B_α.

LEMMA 3.2: *For each $\alpha \in \mathbb{R}$, B_α is a bounded set. Moreover, $B_\alpha = \emptyset$ for each $\alpha > -\inf\{g(\omega, x) \mid x \in \mathbb{R}, \omega \in \Omega\}$.*

Proof: First of all we check that if condition (a.3) of Assumption 3.1 holds, then $\lim_{x\to\pm\infty} g(\omega, x) = \infty$ uniformly in $\omega \in \Omega$.

Let us fix $\omega_1 \in \Omega$. There exist $x_1 \in \mathbb{R}$ and $\delta_1 > 0$ with $\partial g/\partial x(\omega_1, x_1) > \delta_1 > 0$. Thus, by continuity, we can find a neighborhood $V(\omega_1)$ of ω_1 such that $\partial g/\partial x(\omega, x_1) \geq \delta_1 > 0$ for each $\omega \in V(\omega_1)$. Moreover, from condition (a.1) of Assumption 3.1, $\partial g/\partial x$ is an increasing function in the x component, and hence $\partial g/\partial x(\omega, x) \geq \delta_1 > 0$ for each $\omega \in V(\omega_1)$ and $x \geq x_1$. By compactness of Ω, there is a finite covering of $\Omega = \bigcup_{\omega_1\in\Omega} V(\omega_1)$, and we obtain $x_0 \in \mathbb{R}$ and $\delta_0 > 0$ such that $\partial g/\partial x(\omega, x) \geq \delta_0 > 0$ for each $\omega \in \Omega$ and $x \geq x_0$. Similar arguments are valid as x goes to $-\infty$ and we obtain the uniform convergence on Ω, as stated above.

Consequently, if we fix $\alpha \in \mathbb{R}$, we can find $r \in \mathbb{R}$ such that $g(\omega, x) > -\alpha$ for each $x \in \mathbb{R}$ with $|x| > r$ and every $\omega \in \Omega$. Therefore, if $\omega \in \Omega$ and $x_0 > r$, then

$\lim_{t\to t_+} x(t,\omega,x_0,\alpha) = \infty$. Analogously, $\lim_{t\to t_-} x(t,\omega,x_0,\alpha) = -\infty$ whenever $x_0 < -r$ and $\omega \in \Omega$, and B_α is always bounded. Finally, condition (a.3) provides that g is bounded below, and the last assertion of the lemma is immediate. ∎

Let $\alpha \in \mathbb{R}$ such that $B_\alpha \neq \emptyset$. Since the trajectories of B_α are uniformly bounded we deduce that B_α is closed and hence a compact invariant set. We can define

$$x_1(\omega,\alpha) = \inf\{x \mid (\omega,x) \in B_\alpha\}, \quad x_2(\omega,\alpha) = \sup\{x \mid (\omega,x) \in B_\alpha\}.$$

The next result characterizes the invariant set B_α depending on the value of α.

THEOREM 3.3: *Let J be the set of real numbers $\alpha \in \mathbb{R}$ such that $B_\alpha \neq \emptyset$ and contains two hyperbolic minimal subsets, which are both copies of the base. For each $\alpha \in J$, we have*

$$B_\alpha = \{(\omega,x) \in \Omega \times \mathbb{R} \mid x_1(\omega,\alpha) \le x \le x_2(\omega,\alpha)\}. \tag{3.2}$$

Let $\alpha^* = \sup J$. Then

(i) *If $\alpha < 0$ and $|\alpha|$ is large enough, then $\alpha \in J$. Moreover, J is an open interval, i.e., $J = (-\infty, \alpha^*)$.*

(ii) *Let α_1, $\alpha_2 \in J$ with $\alpha_1 < \alpha_2$. Then*

$$x_1(\omega,\alpha_1) < x_1(\omega,\alpha_2) \quad \text{and} \quad x_2(\omega,\alpha_2) < x_2(\omega,\alpha_1)$$

for each $\omega \in \Omega$.

(iii) *The set B_{α^*} is nonempty and we can represent it as*

$$B_{\alpha^*} = \{(\omega,x) \in \Omega \times \mathbb{R} \mid x_1(\omega,\alpha^*) \le x \le x_2(\omega,\alpha^*)\}.$$

Moreover, for each $\omega \in \Omega$

$$x_1(\omega,\alpha^*) = \lim_{\alpha\to(\alpha^*)^-} x_1(\omega,\alpha), \quad x_2(\omega,\alpha^*) = \lim_{\alpha\to(\alpha^*)^-} x_2(\omega,\alpha),$$

and there is a residual invariant set $\Omega_0 \subset \Omega$ such that $x_1(\omega,\alpha^*) = x_2(\omega,\alpha^*)$ for each $\omega \in \Omega_0$.

(iv) *For each $\alpha > \alpha^*$, $B_\alpha = \emptyset$.*

Proof: Assume $\alpha \in J$. Since B_α is compact, for each $\omega \in \Omega$, $i = 1,2$ one has that $(\omega, x_i(\omega,\alpha)) \in B_\alpha$ and we can represent B_α as in (3.2). From Lemma 3.2 we deduce that $\alpha^* \le -\inf\{g(\omega,x) \mid x \in \mathbb{R}, \omega \in \Omega\} < \infty$.

(i) and (ii). For each $r > 0$ we take

$$c_r = \inf\{g(\omega, x) \mid |x| = r, \omega \in \Omega\}, \quad C_r = \sup\{g(\omega, x) \mid |x| = r, \omega \in \Omega\}.$$

We know that $c_r \leq C_r$ and $\lim_{r\to\infty} c_r = \infty$. Take $r > 0$, $\alpha < 0$ with $|\alpha|$ large enough so that $\alpha < -C_r$. There is $r' > r$ such that $-\alpha < c_{r'}$. We consider the following subsets of $\Omega \times \mathbb{R}$

$$K_1 = \{(\omega, x) \in \Omega \times \mathbb{R} \mid -r' \leq x \leq -r\}, \quad K_2 = \{(\omega, x) \in \Omega \times \mathbb{R} \mid r \leq x \leq r'\}.$$

It is easy to check that K_1 is a positively invariant set and K_2 a negatively invariant one. Consequently, there are minimal sets $M_1 \subset K_1$ and $M_2 \subset K_2$, which in particular implies that $x_1(\omega, \alpha) \neq x_2(\omega, \alpha)$ for each $\omega \in \Omega$. Under this condition, Theorem 4.1 of [1] asserts that either the equation $(3.1)_\alpha$ is linear in B_α, which is impossible because $g(\omega_0, x)$ is a strictly convex function in x for some $\omega_0 \in \Omega$, or

$$M_{i,\alpha} = \{(\omega, x_i(\omega, \alpha)) \mid \omega \in \Omega\}, \quad i = 1, 2,$$

are the only minimal sets of B_α and they are hyperbolic, that is, $\alpha \in J$. Moreover, $\lim_{\alpha\to-\infty} x_1(\omega, \alpha) = -\infty$ and $\lim_{\alpha\to-\infty} x_2(\omega, \alpha) = +\infty$.

From the classical results of comparison of solutions, for each $(\omega, x_0) \in \Omega \times \mathbb{R}$, $x(t, \omega, x_0, \alpha)$ is an increasing (resp. decreasing) function in α for $t > 0$ (resp. $t < 0$).

Let $\alpha_2 \in J$ and $\alpha_1 < \alpha_2$; we claim that $\alpha_1 \in J$. We take $\alpha_0 < 0$ with $|\alpha_0|$ large enough in order to get $\alpha_0 \in J$, $\alpha_0 < \alpha_1$ and $B_{\alpha_2} \subset B_{\alpha_0}$. Let $(\omega, x_0) \in B_{\alpha_2}$, that is,

$$x_1(\omega, \alpha_2) \leq x_0 \leq x_2(\omega, \alpha_2).$$

Therefore, for each $t > 0$ we obtain

$$x_1(\omega \cdot t, \alpha_0) \leq x(t, \omega, x_0, \alpha_0) < x(t, \omega, x_0, \alpha_1) < x(t, \omega, x_0, \alpha_2) \leq x_2(\omega \cdot t, \alpha_2),$$

and for each $t < 0$

$$x_1(\omega \cdot t, \alpha_2) \leq x(t, \omega, x_0, \alpha_2) < x(t, \omega, x_0, \alpha_1) < x(t, \omega, x_0, \alpha_0) \leq x_2(\omega \cdot t, \alpha_0).$$

Consequently, $\sup_{t\in\mathbb{R}} |x(t, \omega, x_0, \alpha_1)| < \infty$, the set B_{α_1} is nonempty and $B_{\alpha_2} \subset B_{\alpha_1}$. Thus, $\alpha_1 \in J$ and J is an interval. Moreover, the previous inclusion implies that for each $\omega \in \Omega$

$$x_1(\omega, \alpha_1) < x_1(\omega, \alpha_2) < x_2(\omega, \alpha_2) < x_2(\omega, \alpha_1)$$

and (ii) is proved. Finally, for each $\alpha \in J$, $x_1(\omega,\alpha)$ and $x_2(\omega,\alpha)$ are hyperbolic solutions of $(3.1)_\alpha$; then, Theorem 2.1 provides the same result in a neighborhood of α, which implies that J is an open interval. Hence $J=(-\infty,\alpha^*)$.

(iii) We have shown that $x_1(\omega,\alpha)$ (resp. $x_2(\omega,\alpha)$) is an increasing (resp. decreasing) function in α. Therefore, there exist the limits

$$x_1^*(\omega)=\lim_{\alpha\to(\alpha^*)^-} x_1(\omega,\alpha), \quad x_2^*(\omega)=\lim_{\alpha\to(\alpha^*)^-} x_2(\omega,\alpha).$$

It is also easy to check that, for each $\omega\in\Omega$ and $t\in\mathbb{R}$

$$x_1^*(\omega\cdot t)=x(t,\omega,x_1^*(\omega),\alpha^*), \quad x_2^*(\omega\cdot t)=x(t,\omega,x_2^*(\omega),\alpha^*),$$

which implies that $\sup_{t\in\mathbb{R}}|x(t,\omega,x_i^*(\omega),\alpha^*)|<\infty$ for $i=1,2$. Consequently, the set B_{α^*} is nonempty and

$$x_1(\omega,\alpha)\le x_1(\omega,\alpha^*)\le x_1^*(\omega)\le x_2^*(\omega)\le x_2(\omega,\alpha^*)\le x_2(\omega,\alpha)$$

for each $\omega\in\Omega$ and $\alpha<\alpha^*$. Taking limits as α tends to α^* from below, we conclude that $x_1^*(\omega)=x_1(\omega,\alpha^*)$ and $x_2^*(\omega)=x_2(\omega,\alpha^*)$, as asserted.

Let us assume that $x_1(\omega,\alpha^*)\ne x_2(\omega,\alpha^*)$ for each $\omega\in\Omega$. Then we could apply again Theorem 4.1 of [1] to conclude that $\alpha^*\in J$, which is impossible. Therefore, there is $\omega_0\in\Omega$ with $x_1(\omega_0,\alpha^*)=x_2(\omega_0,\alpha^*)$ which also implies that $x_1(\omega_0\cdot t,\alpha^*)=x_2(\omega_0\cdot t,\alpha^*)$ for each $t\in\mathbb{R}$ because

$$x_i(\omega_0\cdot t,\alpha^*)=x(t,\omega_0,x_i(\omega_0,\alpha^*),\alpha^*), \quad i=1,2. \tag{3.3}$$

Now Theorem 5.1 of [1] asserts that there exists a residual invariant subset $\Omega_0\subset\Omega$ such that $x_1(\omega,\alpha^*)=x_2(\omega,\alpha^*)$ for each $\omega\in\Omega_0$. The ergodic and topological structure of the set B_{α^*} will be described in the next proposition.

(iv) Let $\alpha>\alpha^*$. If $x_0>x_2(\omega,\alpha^*)$, the point $(\omega,x_0)\notin B_\alpha$ because for $t>0$, $x(t,\omega,x_0,\alpha)>x(t,\omega,x_0,\alpha^*)$ and $x(t,\omega,x_0,\alpha^*)$ tends to ∞ as $t\to\infty$. Analogously, if $x_0<x_1(\omega,\alpha^*)$, the point $(\omega,x_0)\notin B_\alpha$, and we conclude that $B_\alpha\subset B_{\alpha^*}$.

Let us assume that B_α is nonempty and then takes the form (3.2). We consider $(\omega_0,x_0)\in B_{\alpha^*}$ with $x_1(\omega_0,\alpha^*)=x_2(\omega_0,\alpha^*)$. From $B_\alpha\subset B_{\alpha^*}$ we deduce that

$$x_1(\omega_0,\alpha)=x_2(\omega_0,\alpha)=x_1(\omega_0,\alpha^*)=x_2(\omega_0,\alpha^*).$$

This implies that for each $t\in\mathbb{R}$

$$x_1(\omega_0\cdot t,\alpha)=x_2(\omega_0\cdot t,\alpha)=x_1(\omega_0\cdot t,\alpha^*)=x_2(\omega_0\cdot t,\alpha^*),$$

which is impossible because from relation (3.3) and the analogous one for α, they are solutions of different differential equations, and the proof is complete. ∎

Let us fix an ergodic measure m on Ω. Denote by $\gamma_i(\alpha)$ the Lyapunov exponents with respect to m given by the formula

$$\gamma_i(\alpha) = \int_\Omega \frac{\partial g}{\partial x}(\omega, x_i(\omega,\alpha))dm, \quad i = 1, 2.$$

We know that $\gamma_1(\alpha) < 0 < \gamma_2(\alpha)$ for each $\alpha \in J$.

If the base flow (Ω,σ) is almost periodic, almost automorphic dynamics, i.e., the presence of an almost automorphic minimal set which is not almost periodic, can only occur in a few situations, all of them for the unique value α^* of the parameter. The ergodic and topological structure of B_{α^*} is described in the following result.

PROPOSITION 3.4: *Let α^* be the value of the parameter obtained in Theorem 3.3. Then, there exists a unique minimal subset $M_{\alpha^*} \subset B_{\alpha^*}$, which is an almost automorphic extension of the base (Ω,σ) and it is not hyperbolic. Besides, one of the following cases holds.*

(c.1) $x_1(\omega,\alpha^*) = x_2(\omega,\alpha^*)$ *for every $\omega \in \Omega$. In this situation $B_{\alpha^*} = M_{\alpha^*}$ is a copy of the base, there exists a unique ergodic measure concentrated on B_{α^*} and projecting onto m, and the Lyapunov exponent is null. If the base (Ω,σ) is a.p., then so is M_{α^*}.*

(c.2) $x_1(\omega,\alpha^*) = x_2(\omega,\alpha^*)$ *for almost every $\omega \in \Omega$ with respect to m. As in the previous case, there exists a unique ergodic measure concentrated on B_{α^*} and projecting onto m, and the Lyapunov exponent is null. If (Ω,σ) is a.p., then M_{α^*} is an a.a. minimal set, which is a.p. if and only if it is a copy of the base.*

(c.3) $x_1(\omega,\alpha^*) = x_2(\omega,\alpha^*)$ *only on a set of null measure. Then, there exist two unique ergodic measures concentrated on B_{α^*} and projecting onto m, and the Lyapunov exponents are not zero: $\gamma_1(\alpha^*) < 0 < \gamma_2(\alpha^*)$. In this case if (Ω,σ) is a.p., then M_{α^*} is an a.a. minimal set which is not a.p.*

Proof: From (iii) of Theorem 3.3, we deduce that

$$\gamma_i(\alpha^*) = \lim_{\alpha\to(\alpha^*)^-} \gamma_i(\alpha), \quad i = 1, 2,$$

which implies that $\gamma_1(\alpha^*) \le 0 \le \gamma_2(\alpha^*)$. The rest of the proof is an application of the results stated in [1, Section 5], taking into account that, as we have mentioned before, the equation cannot be linear in B_{α^*} because $g(\omega_0, x)$ is a strictly convex function in x. In particular, if M_{α^*} was hyperbolic, the maps $x_i(\omega, \alpha^*)$, $i = 1, 2$, would coincide identically by Theorem 5.1 of [1]. This would imply that we are in case (c.1) and the Lyapunov exponent is null, which contradicts the hyperbolicity of M_{α^*}. ∎

Next we show that we can weaken the hypotheses given in Assumption 3.1 obtaining the same results, that is, the existence of a value α^* with the property that there may exist an almost automorphic and not almost periodic minimal subset. More precisely, we will only require the convexity properties for the function g over a compact set of bounded trajectories with nonempty interior.

PROPOSITION 3.5: *Let B be a set of bounded trajectories for the family of equations $x' = h(\omega{\cdot}t, x)$, $\omega \in \Omega$, where $h\colon \Omega\times\mathbb{R} \to \mathbb{R}$ is continuous, differentiable with respect to x and $\partial h/\partial x\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous. We assume that*

(b.1) *B is a compact invariant set which can be represented as*

$$B = \{(\omega, x) \in \Omega \times \mathbb{R} \mid x_1(\omega) \le x \le x_2(\omega)\},$$

with $x_1(\omega) \ne x_2(\omega)$ for each $\omega \in \Omega$;

(b.2) *when restricted to B, the function h is a convex map in x for each $\omega \in \Omega$, and strictly convex in x for some $\omega_0 \in \Omega$.*

Let J be the set of positive real numbers $\alpha > 0$ such that there is a nonempty compact invariant subset $\widehat{B}_\alpha \subset B$ of bounded solutions for the equation $x' = h(\omega \cdot t, x) + \alpha$, which contains two hyperbolic minimal subsets and admits a representation

$$\widehat{B}_\alpha = \{(\omega, x) \in \Omega \times \mathbb{R} \mid x_1(\omega, \alpha) \le x \le x_2(\omega, \alpha)\}. \tag{3.4}$$

Let $\alpha^* = \sup J$. Then

(i) *J is an open interval, i.e., $J = (0, \alpha^*)$.*

(ii) *Let α_1, $\alpha_2 \in J$ with $\alpha_1 < \alpha_2$. Then*

$$x_1(\omega, \alpha_1) < x_1(\omega, \alpha_2) \quad \textit{and} \quad x_2(\omega, \alpha_2) < x_2(\omega, \alpha_1)$$

for each $\omega \in \Omega$.

(iii) *For each $\omega \in \Omega$, there exist the limits*

$$x_1(\omega, \alpha^*) = \lim_{\alpha \to (\alpha^*)^-} x_1(\omega, \alpha), \quad x_2(\omega, \alpha^*) = \lim_{\alpha \to (\alpha^*)^-} x_2(\omega, \alpha).$$

Besides, the nonempty invariant set $\widehat{B}_{\alpha^*}$ defined as

$$\widehat{B}_{\alpha^*} = \{(\omega, x) \in \Omega \times \mathbb{R} \mid x_1(\omega, \alpha^*) \le x \le x_2(\omega, \alpha^*)\}$$

contains bounded solutions for the equation $x' = h(\omega\cdot t, x) + \alpha^*$, and there is a residual invariant set $\Omega_0 \subset \Omega$ such that $x_1(\omega, \alpha^*) = x_2(\omega, \alpha^*)$ for each $\omega \in \Omega_0$.

Moreover, the same conclusions of Proposition 3.4 hold in this case for $\widehat{B}_{\alpha^*}$, making the obvious changes.

Proof: The main idea of the proof is the construction of a continuous function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, which satisfies Assumption 3.1 and coincides with $h(\omega, x)$ when $(\omega, x) \in B$. It is easy to check that the function

$$g(\omega, x) = h(\omega, x_2(\omega)) + \int_{x_2(\omega)}^{x} f(\omega, s)ds,$$

where

$$f(\omega, x) = \begin{cases} \frac{\partial h}{\partial x}(\omega, x_1(\omega)) + x - x_1(\omega) & \text{if } x \le x_1(\omega), \\ \frac{\partial h}{\partial x}(\omega, x) & \text{if } x \in [x_1(\omega), x_2(\omega)], \\ \frac{\partial h}{\partial x}(\omega, x_2(\omega)) + x - x_2(\omega) & \text{if } x \ge x_2(\omega), \end{cases}$$

satisfies our requirements. Moreover, from Theorem 4.1 of [1], B coincides with B_0, the set of bounded trajectories for $x' = g(\omega \cdot t, x)$, because the equation is not linear and then $x_1(\omega)$ and $x_2(\omega)$ are two hyperbolic solutions.

Finally, all the statements follow easily from the application of Theorem 3.3 to the new family of equations $x' = g(\omega \cdot t, x) + \alpha$, because $B_\alpha \subset B_0 = B$ for each $\alpha > 0$, and we can take $\widehat{B}_\alpha = B_\alpha$. Notice that in this case, the original family of equations $x' = h(\omega \cdot t, x) + \alpha$ may have more bounded solutions outside B_α. ∎

4. Continuous dependence

Let $g\colon \mathbb{R} \to \mathbb{R}$ be a real function satisfying the corresponding Assumption 3.1. Notice that in this case, (a.2) means that g is a strictly convex function. For each real and continuous function p we consider the scalar differential equation

$$x' = g(x) + p(t), \quad t \in \mathbb{R}. \tag{4.1}$$

We denote by $x(t, x_0, p)$ the solution of (4.1) with initial condition x_0 and let $B(p) = \{x_0 \in \mathbb{R} \mid \sup_{t\in\mathbb{R}} |x(t, x_0, p)| < \infty\}$ be the set of bounded solutions for equation (4.1).

PROPOSITION 4.1: *Let $\{p_j(t)\}_{j\in\mathbb{N}}$ be a sequence of continuous and bounded real functions satisfying*

(i) *there is a continuous and bounded real function p such that $\lim_{j\to\infty} p_j = p$ in the weak* topology $\sigma(L^\infty(\mathbb{R}), L^1(\mathbb{R}))$, and*

(ii) *$B(p_j) \neq \emptyset$ for each $j \in \mathbb{N}$.*

Then equation (4.1) has at least one bounded solution, i.e., $B(p) \neq \emptyset$.

Proof: From the uniform boundedness principle, every weakly* convergent sequence is norm bounded. Then, there is an $M > 0$ such that $||p_j||_\infty \leq M$ for every $j \in \mathbb{N}$. From this fact and $\lim_{x\to\pm\infty} g(x) = \infty$, we can find a positive constant $C > 0$ such that $B(p_j) \subset [-C, C]$ for every $j \in \mathbb{N}$. Since $B(p_j) \neq \emptyset$ for each $j \in \mathbb{N}$, we take a sequence $\{x_j\}_{j\in\mathbb{N}}$ with $x_j \in B(p_j)$ for each $j \in \mathbb{N}$. Thus, $|x_j| \leq C$ for every $j \in \mathbb{N}$, and there is a convergent subsequence. Let us assume that the whole sequence converges to some point x_0.

We consider the family of solutions $\{x(t, x_j, p_j)\}_{j\in\mathbb{N}}$, which is equicontinuous and uniformly bounded. Applying the Arzelà–Ascoli theorem and a Cantor diagonalization process we obtain a subsequence $\{x(t, x_{j_k}, p_{j_k})\}_{k\in\mathbb{N}}$ which converges uniformly on the compact sets of $\mathbb{R}$ to a function $x_0(t)$. Obviously, $|x_0(t)| \leq C$ for any $t \in \mathbb{R}$, so that it is bounded. Finally, from the weak* convergence of p_j to p applied to the function $\chi_{[0,t]} \in L^1(\mathbb{R})$, we obtain that $\int_0^t p_{j_k}(s)ds \to \int_0^t p(s)ds$, for each $t \in \mathbb{R}$. From this, it is easy to check that $x_0(t)$ coincides with $x(t, x_0, p)$, the solution of (4.1) with initial value x_0, and the proof is finished. ∎

As in the previous section, (Ω, σ) is a minimal flow defined on a compact metric space. For each real and continuous function $p \in C(\Omega)$ and $\alpha \in \mathbb{R}$, we consider the family of differential equations

$$x' = g(x) + p(\omega \cdot t) + \alpha, \quad \omega \in \Omega, \tag{4.2$_\alpha$}$$

where g satisfies Assumption 3.1. As in Section 3, we denote by $\alpha^*(p)$ the real number obtained in Theorem 3.3 for the corresponding family $(4.2)_\alpha$.

PROPOSITION 4.2: *Let $p, q \in C(\Omega)$. Then*

$$|\alpha^*(q) - \alpha^*(p)| \leq ||q - p||_\infty.$$

Proof: We can assume that $\alpha^*(p) \leq \alpha^*(q)$; the opposite situation is analogous. We claim that if $\alpha = \alpha^*(q) - ||p - q||_\infty$, then equation $(4.2)_\alpha$ has bounded solutions, that is, $\alpha^*(q) - ||p - q||_\infty \leq \alpha^*(p)$, as stated.

We denote by $x_0(t, \omega)$ a bounded solution of $x' = g(x) + q(\omega \cdot t) + \alpha^*(q)$, $\omega \in \Omega$, and by $x(t, \omega)$ the solution of $(4.2)_\alpha$ such that $x(0, \omega) = x_0(0, \omega)$. Then, from

$$x' = g(x) + p(\omega \cdot t) + \alpha \le g(x) + q(\omega \cdot t) + \alpha^*(q),$$

the classical comparison theorems show that $x(t,\omega) \le x_0(t,\omega)$ for each $t \ge 0$ and $x_0(t,\omega) \le x(t,\omega)$ for each $t \le 0$. From these inequalities, the boundedness of $x_0(t,\omega)$ and the behaviour of the solutions induced by the assumption $\lim_{x\to\pm\infty} g(x) = \infty$, it is easy to check that $x(t,\omega)$ is also bounded, which finishes the proof, as explained above. ∎

The above result implies the continuity of α^* in $C(\Omega)$ for the topology of uniform convergence, as we state in the next result. We also show a kind of semicontinuity result for the weak topology $\sigma(C(\Omega), M(\Omega))$.

PROPOSITION 4.3: *Let $\{p_j\}_{j\in\mathbb{N}}$ be a sequence of real continuous functions on Ω.*

(i) *If there is an $\omega_0 \in \Omega$ and $p \in C(\Omega)$ such that $\lim_{j\to\infty} p_j(\omega_0 \cdot t) = p(\omega_0 \cdot t)$ in the weak* $\sigma(L^\infty(\mathbb{R}), L^1(\mathbb{R}))$-topology, then $\limsup_{j\to\infty} \alpha^*(p_j) \le \alpha^*(p)$.*
(ii) *Let us assume that $\lim_{j\to\infty} p_j = p$ in the weak $\sigma(C(\Omega), M(\Omega))$-topology; then $\limsup_{j\to\infty} \alpha^*(p_j) \le \alpha^*(p)$.*
(iii) *If $\lim_{j\to\infty} p_j = p$ uniformly on Ω, then $\lim_{j\to\infty} \alpha^*(p_j) = \alpha^*(p)$.*

Proof: (i) Let us assume that for a subsequence $\lim_{k\to\infty} \alpha^*(p_{j_k}) = \alpha_0$ and let us prove that $\alpha_0 \le \alpha^*(p)$. Recall that by the minimal character of the base flow (Ω, σ) the value $\alpha^*(p)$ is independent of ω. Therefore, we apply Proposition 4.1 to $p_{j_k}(\omega_0 \cdot t) + \alpha^*(p_{j_k})$, which converge weakly* to $p(\omega_0 \cdot t) + \alpha_0$, to obtain that $x' = g(x) + p(\omega_0 \cdot t) + \alpha_0$ admits a bounded solution, that is, $\alpha_0 \le \alpha^*(p)$. As a consequence $\limsup_{j\to\infty} \alpha^*(p_j) \le \alpha^*(p)$, and (i) is proved.

(ii) It is a consequence of (i) because now we prove that for each $\omega \in \Omega$, $p_j(\omega \cdot t)$ converges to $p(\omega \cdot t)$ in the weak* $\sigma(L^\infty(\mathbb{R}), L^1(\mathbb{R}))$-topology. We have to check that $\lim_{j\to\infty} \int_{\mathbb{R}} p_j(\omega \cdot t)\varphi(t)dt = \int_{\mathbb{R}} p(\omega \cdot t)\varphi(t)dt$, for each $\varphi \in L^1(\mathbb{R})$. However, if we fix $\omega \in \Omega$ and $\varphi \in L^1(\mathbb{R})$, from Riesz representation theorem there is a $\mu \in M(\Omega)$ such that $\int_{\mathbb{R}} q(\omega \cdot t)\varphi(t)dt = \int_\Omega q\,d\mu$ for every $q \in C(\Omega)$, and the claim follows from the weak $\sigma(C(\Omega), M(\Omega))$ convergence.

(iii) It is immediate from Proposition 4.2. ∎

Next we will consider the family of systems $(4.2)_\alpha$ with $\alpha = 0$. For simplicity of notation we will remove α from every set instead of substituting it by 0. Thus, as in Section 3, we will consider the sets of bounded solutions $B(p)$, and in the case they are nonempty we define

$$x_1(\omega, p) = \inf\{x \mid (\omega, x) \in B(p)\}, \quad x_2(\omega, p) = \sup\{x \mid (\omega, x) \in B(p)\}.$$

The next result gives a characterization of the convergence of the Lyapunov exponents in terms of the convergence in measure of these functions. We fix an ergodic measure m on Ω.

THEOREM 4.4: *Let $\{p_j\}_{j\in\mathbb{N}}$ be a sequence of real continuous functions on Ω such that p_j converges to p in the $\sigma(C(\Omega), M(\Omega)))$-topology, and $B(p_j)$ is nonempty for each $j \in \mathbb{N}$. Denote by $\gamma_i(p)$, $i = 1, 2$, the Lyapunov exponents with respect to m of the systems (4.2)$_\alpha$ for $\alpha = 0$. The following statements are equivalent:*

(i) $\lim_{j\to\infty} \gamma_i(p_j) = \gamma_i(p)$ *for* $i = 1, 2$.

(ii) $\lim_{j\to\infty} x_i(\omega, p_j) = x_i(\omega, p)$ *in measure for* $i = 1, 2$.

Proof: Every weakly convergent sequence is norm bounded, thus, there is an $M > 0$ such that $\sup_{j\in\mathbb{N}} ||p_j||_\infty < M$. From this fact and $\lim_{x\to\pm\infty} g(x) = \infty$, we can find a positive constant $C > 0$ such that $B(p)$, $B(p_j) \subset \Omega \times [-C, C]$ for every $j \in \mathbb{N}$. In particular, $-C \le x_i(\omega, p_j)$, $x_i(\omega, p) \le C$, for each $\omega \in \Omega$, $j \in \mathbb{N}$ and $i = 1, 2$.

(ii) $\Rightarrow$ (i) The convergence in measure implies the existence of a subsequence converging almost everywhere with respect to m. Therefore, there is a subsequence $\{x_i(\omega, p_{j_k})\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_i(\omega, p_{j_k}) = x_i(\omega, p)$ a.e. for each $i = 1, 2$. Thus,

$$\lim_{k\to\infty} \gamma_i(p_{j_k}) = \lim_{k\to\infty} \int_\Omega g'(x_i(\omega, p_{j_k}))dm = \int_\Omega g'(x_i(\omega, p))dm = \gamma_i(p)$$

for each $i = 1, 2$, and the result also holds immediately for the whole sequence.

(i) $\Rightarrow$ (ii) For each $j \in \mathbb{N}$, let ν_j be the ergodic measure concentrated into the subset $M_2(p_j) = \{(\omega, x_2(\omega, p_j)) \mid \omega \in \Omega\}$. All these ergodic measures ν_j are concentrated in the compact subset $K_C = \Omega \times [-C, C]$, from which we deduce the existence of a weakly convergent subsequence (assume it is the whole sequence), i.e., there is a measure ν such that

$$\lim_{j\to\infty} \int_{K_C} f d\nu_j = \int_{K_C} f d\nu \tag{4.3}$$

for each continuous function $f \in C(K_C)$. The family of normalized measures $(\nu_\omega)_{\omega\in\Omega}$ denotes the disintegration of ν with respect to m.

From Proposition 4.1 we know that $B(p) \ne \emptyset$. Next we check that $\nu(B(p)) = 1$. It is well known that

$$\nu(B(p)) = \inf\left\{ \int_{K_C} f d\nu \mid f \in C(K_C) \text{ with } f_{|B(p)} \equiv 1 \text{ and } 0 \le f \le 1 \right\}.$$

Take any $f \in C(K_C)$ with $f_{|B(p)} \equiv 1$ and $0 \leq f \leq 1$ and let us see that $\lim_{j\to\infty} f(\omega, x_2(\omega, p_j)) = 1$ pointwise. For each $\omega \in \Omega$ there exists a subsequence $\{j_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_2(\omega, p_{j_k}) = x_0$ for some x_0. Arguing as in Proposition 4.1 we conclude that $(\omega, x_0) \in B(p)$. The continuity of the function f provides $\lim_{j\to\infty} f(\omega, x_2(\omega, p_{j_k})) = f(\omega, x_0) = 1$. As for any subsequence there is another one through which the limit is 1, we are done. Hence, the Lebesgue theorem leads to

$$1 = \lim_{j\to\infty} \int_\Omega f(\omega, x_2(\omega, p_j))dm = \lim_{j\to\infty} \int_{K_C} f d\nu_j = \int_{K_C} f d\nu,$$

and then, $\nu(B(p)) = 1$, as claimed.

From 4.3 and the continuity of g' we deduce that

$$\begin{aligned}\lim_{j\to\infty} \gamma_2(p_j) &= \lim_{j\to\infty} \int_\Omega g'(x_2(\omega, p_j))dm = \lim_{j\to\infty} \int_{K_C} g' d\nu_j \\ &= \int_{K_C} g' d\nu = \int_\Omega \left[\int_{x_1(\omega,p)}^{x_2(\omega,p)} g'(u) d\nu_\omega \right] dm.\end{aligned}$$

Moreover, from $\lim_{j\to\infty} \gamma_2(p_j) = \gamma_2(p) = \int_\Omega g'(x_2(\omega, p))dm$ we obtain

$$\int_\Omega \left[g'(x_2(\omega, p)) - \int_{x_1(\omega,p)}^{x_2(\omega,p)} g'(u) d\nu_\omega \right] dm = 0.$$

The strictly increasing character of g' allows us to assert

$$\int_{x_1(\omega,p)}^{x_2(\omega,p)} g'(u) d\nu_\omega \leq \int_{x_1(\omega,p)}^{x_2(\omega,p)} g'(x_2(\omega, p)) d\nu_\omega \leq g'(x_2(\omega, p)),$$

from which it follows that for almost every ω with respect to m

$$\int_{x_1(\omega,p)}^{x_2(\omega,p)} g'(u) d\nu_\omega = g'(x_2(\omega, p)), \text{ i.e., } \int_{x_1(\omega,p)}^{x_2(\omega,p)} [g'(u) - g'(x_2(\omega, p))] d\nu_\omega = 0.$$

Finally, since $g'(x_2(\omega, p)) - g'(u) > 0$ whenever $x_1(\omega, p) < u < x_2(\omega, p)$, we conclude that ν is concentrated into $M_2(p) = \{(\omega, x_2(\omega, p)) \mid \omega \in \Omega\}$. Consequently, the weak convergence 4.3 means that for each $f \in C(K_C)$

$$\lim_{j\to\infty} \int_\Omega f(\omega, x_2(\omega, p_j))dm = \int_\Omega f(\omega, x_2(\omega, p))dm, \tag{4.4}$$

and we deduce that $\lim_{j\to\infty} \|x_2(\cdot, p_j)\|_2 = \|x_2(\cdot, p)\|_2$ for $f(\omega, x) = x^2$.

Next we check that the sequence of functions $x_2(\omega, p_j)$ converge to $x_2(\omega, p)$ as $j \to \infty$ in the $\sigma(L^2(\Omega), L^2(\Omega))$-topology. We have to prove that for each $y \in L^2(\Omega)$

$$\lim_{j\to\infty} \int_\Omega y(\omega) x_2(\omega, p_j)\, dm = \int_\Omega y(\omega) x_2(\omega, p)\, dm.$$

From the density of the continuous functions in $L^2(\Omega)$ we know that given $\varepsilon > 0$ there is $h \in C(\Omega)$ such that $\|y - h\|_2 < \varepsilon$. Moreover,

$$\begin{aligned} \left| \int_\Omega y(\omega)[x_2(\omega, p_j) - x_2(\omega, p)]\, dm \right| \leq & \int_\Omega |y(\omega) - h(\omega)| |x_2(\omega, p_j) - x_2(\omega, p)|\, dm \\ & + \left| \int_\Omega h(\omega)[x_2(\omega, p_j) - x_2(\omega, p)]\, dm \right| = I_1^j + I_2^j. \end{aligned}$$

However, from Cauchy–Schwarz inequality and the uniform bound for the functions $x_2(\omega, p_j)$, there is a positive constant $K > 0$ such that

$$I_1^j \leq \|y - h\|_2 \|x_2(\cdot, p_j) - x_2(\cdot, p)\|_2 < K\varepsilon.$$

In addition, I_2^j goes to 0 as $j \to \infty$ by relation 4.4 with $f(\omega, x) = h(\omega)x$, and the weak $\sigma(L^2(\Omega), L^2(\Omega))$-convergence is obtained.

Finally, from the $\sigma(L^2(\Omega), L^2(\Omega))$-convergence and the above convergence of the norms we deduce that $\lim_{j\to\infty} x_2(\omega, p_j) = x_2(\omega, p)$ in the L^2-topology, which in particular implies the convergence in measure of $x_2(\omega, p_j)$ to $x_2(\omega, p)$ as $j \to \infty$. A similar proof applies to $x_1(\omega, p)$ and the theorem is proved. ∎

To finish this section, we give a kind of semicontinuity result for the Lyapunov exponents when the weak topology is considered. We maintain the notation of the previous theorem.

Proposition 4.5: *Under the same assumptions of Theorem 4.4,*

$$\gamma_1(p) \leq \liminf_{j\to\infty} \gamma_1(p_j) \quad \textit{and} \quad \limsup_{j\to\infty} \gamma_2(p_j) \leq \gamma_2(p).$$

Proof: We prove it for $\gamma_2(p_j)$. Suppose that $\lim_{k\to\infty} \gamma_2(p_{j_k}) = \gamma$ and let us check that $\gamma \leq \gamma_2(p)$. As in the previous proof, and taking into account that g' is a strictly increasing function, we obtain

$$\gamma = \lim_{k\to\infty} \gamma_2(p_{j_k}) = \int_\Omega \left[\int_{x_1(\omega,p)}^{x_2(\omega,p)} g'(u)\, d\nu_\omega \right] dm \leq \int_\Omega g'(x_2(\omega, p))\, dm = \gamma_2(p),$$

as asserted. The proof for $\gamma_1(p_j)$ is completely analogous. ∎

5. Limit periodic case

We start this section by recalling some notions on almost periodic functions. A continuous real function $f\colon \mathbb{R} \to \mathbb{R}$ is **Bohr almost periodic** if for every $\varepsilon > 0$ the set $E_\varepsilon = \{s \in \mathbb{R} \mid |f(t+s) - f(t)| < \varepsilon \text{ for each } t \in \mathbb{R}\}$ is relatively dense, that is, there exists a positive number $l_\varepsilon > 0$ such that each interval of length l_ε contains at least an element of E_ε. A continuous real function f is said to be **limit periodic** if there exists a sequence of continuous and periodic functions $\{f_n\}_{n\in\mathbb{N}}$ converging uniformly to f. A limit periodic function is always almost periodic.

With any continuous almost periodic function f, one can associate a mean value

$$M(f) = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(s)ds,$$

and a Fourier series $f \sim \sum_{n=1}^{\infty} c_n e^{i\lambda_n t}$, where the frequencies λ_n are the (denumerably many) values of λ for which $M(fe^{-i\lambda t}) \neq 0$, i.e.,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t e^{-i\lambda_n s} f(s)ds \neq 0.$$

The **frequency module** is the set $\mathcal{M}(f) = \{\sum_n j_n\lambda_n \mid j_n \in \mathbb{Z}\}$ of finite integer combinations of these frequencies, that is, the smallest additive subgroup of $\mathbb{R}$ containing all the frequencies. We will denote by $\mathcal{A}(\mathcal{M}(f))$ the set of all almost periodic functions with frequency module contained in $\mathcal{M}(f)$. It is closed in the uniform topology, it is separable (in contrast to the space of almost periodic functions), and it is an algebra: if $f, g \in \mathcal{A}(\mathcal{M}(f))$, then $f \cdot g \in \mathcal{A}(\mathcal{M}(f))$.

Each almost periodic function f is a uniform limit of trigonometric polynomials $\sum_{j=1}^{n} c_j e^{i\alpha_j t}$ with $\alpha_j \in \mathcal{M}(f)$. In particular, if f is limit periodic the above functions are periodic because it can be shown that, in this case, each two elements of $\mathcal{M}(f)$ are commensurable, i.e., linearly dependent over $\mathbb{Q}$.

Let us recall the construction of the **hull** Ω of a real bounded and uniformly continuous function f. It is the closure of the set of mappings $\{f_t \mid t \in \mathbb{R}\}$ where $f_t(s) = f(t+s)$, in the topology of uniform convergence on compact sets, i.e., the compact-open topology. The translation $\mathbb{R} \times \Omega \to \Omega$, $(t,\omega) \mapsto \omega \cdot t$, with $\omega \cdot t(s) = \omega(t+s)$, defines a continuous flow σ on Ω.

If f is an almost periodic function, Ω is a compact topological group and the flow (Ω, σ) is minimal and uniquely ergodic. It can be shown that $\mathcal{A}(\mathcal{M}(f))$ and $C(\Omega)$ are isomorphic as Banach algebras. Then, $h \in \mathcal{A}(\mathcal{M}(f))$ if and only if there exists $q \in C(\Omega)$ such that for every $t \in \mathbb{R}$, $h(t) = q(\omega_0 \cdot t)$ for some $\omega_0 \in \Omega$ (see Johnson and Moser [12] for the details).

Now we explain how to obtain a collective formulation, i.e., a family of equations from a single one by means of the hull. We consider the equation

$$x' = g(x) + p_0(t), \quad t \in \mathbb{R}, \tag{5.1}$$

where g is a real function satisfying the corresponding Assumption 3.1, and $p_0(t)$ is a real bounded and uniformly continuous function.

The function p_0 has a unique extension to a continuous function on its hull Ω_0 given by $p\colon \Omega_0 \to \mathbb{R}$, $\omega \mapsto \omega(0)$. Thus, we can consider the family of equations

$$x' = g(x) + p(\omega \cdot t), \quad \omega \in \Omega_0,$$

which in particular, when $\omega = p_0$, coincides with the initial equation (5.1). Notice that each ω provides a different equation but the value of $\alpha^*(p)$, as defined in Theorem 3.3, is constant for all of them when the flow on Ω_0 is minimal. Therefore, in this case we define $\alpha^*(p_0)$ as $\alpha^*(p)$.

As in Section 3, for any minimal flow (Ω, σ), we denote by $B_\alpha(p)$ the set of bounded solutions for the family $x' = g(x) + p(\omega \cdot t) + \alpha$, $\omega \in \Omega$, which, if nonempty, can be represented as

$$B_\alpha(p) = \{(\omega, x) \in \Omega \times \mathbb{R} \mid x_1(\omega, \alpha) \le x \le x_2(\omega, \alpha)\}.$$

Notice that $x_i(\omega, \alpha)$, $i = 1, 2$, depend on p although, for simplicity, we have dropped it from the notation. We denote by $M_{\alpha^*(p)}$ the minimal set contained in $B_{\alpha^*(p)}(p)$, provided by Proposition 3.4.

When Ω is the hull of a limit periodic function, we will show in Theorem 5.3 that the first situation stated in Proposition 3.4 is generic in $C(\Omega)$, i.e., it happens for p in a residual set of functions of $C(\Omega)$. Before showing this fact, we study the periodic case.

LEMMA 5.1: *Let p_0 be a continuous and periodic real function. Then*

$$x' = g(x) + p_0(t) + \alpha^*(p_0), \quad t \in \mathbb{R}, \tag{5.2}$$

admits a unique bounded solution which is periodic.

Proof: Let Ω_0 be the hull of p_0 and consider, as above, the family of differential equations

$$x' = g(x) + p(\omega \cdot t) + \alpha^*(p_0), \quad \omega \in \Omega_0.$$

Since p_0 is periodic of period T, its hull $\Omega_0 = \{p_0(t+s) \mid 0 \le s \le T\}$ and, as we have said before, $\alpha^*(p_0) = \alpha^*(p)$.

We know that there is $\omega_0 \in \Omega_0$ such that $x_1(\omega_0, \alpha^*(p)) = x_2(\omega_0, \alpha^*(p))$, and then $x_1(\omega_0 {\cdot} t, \alpha^*(p)) = x_2(\omega_0 {\cdot} t, \alpha^*(p))$ for each $t \in \mathbb{R}$. Moreover, for $\omega \in \Omega_0$ there is $s \in \mathbb{R}$ such that $\omega = \omega_0 \cdot s$ and therefore, $x_1(\omega, \alpha^*(p)) = x_2(\omega, \alpha^*(p))$ for each $\omega \in \Omega_0$, i.e., we are in situation (c.1) of Proposition 3.4, and $B_{\alpha^*(p)}(p) = M_{\alpha^*(p)}$. Consequently, if we take $\omega_1 = p_0$, then $x_1(\omega_1 \cdot t, \alpha^*(p))$ is the only bounded solution of (5.2) and it is periodic. ∎

Let (Ω, σ) be a minimal flow. We denote by

$$\mathcal{P}(\Omega) = \{p \in C(\Omega) \mid \text{ there is } T > 0 \text{ with } p(\omega) = p(\omega \cdot T) \text{ for each } \omega \in \Omega\}$$

the set of continuous and periodic functions on Ω. Next we deduce that situation (c.1) of Proposition 3.4 always happens when $p \in \mathcal{P}(\Omega)$.

Corollary 5.2: *Let (Ω, σ) be a minimal flow, and let us assume that $p \in \mathcal{P}(\Omega)$ is a continuous and periodic function. Then $x_1(\omega, \alpha^*(p)) = x_2(\omega, \alpha^*(p))$ for each $\omega \in \Omega$ and $B_{\alpha^*(p)}(p) = M_{\alpha^*(p)}$.*

Proof: It is an application of the previous lemma to each real periodic function $p_\omega(t) = p(\omega \cdot t)$ for $\omega \in \Omega$. ∎

Theorem 5.3: *Let Ω be the hull of a limit periodic function. Then, there exists a residual subset $R \subset C(\Omega)$ such that, for each $p \in R$, $x_1(\omega, \alpha^*(p)) = x_2(\omega, \alpha^*(p))$ for every $\omega \in \Omega$ and $B_{\alpha^*(p)}(p) = M_{\alpha^*(p)}$ is almost periodic.*

Proof: For each $n \in \mathbb{N}$ we consider the subset

$$V_n = \{p \in C(\Omega) \mid \exists \alpha < \alpha^*(p) \text{ with } \|x_2(\cdot, \alpha) - x_1(\cdot, \alpha)\|_\infty < 1/n\}.$$

It is easy to check that V_n is an open set because, as we proved in Theorem 2.1, the hyperbolic character of equation $(4.2)_\alpha$ is maintained in a neighborhood of p. Now we consider the $\mathcal{G}_\delta$ set $R = \bigcap_{n \in \mathbb{N}} V_n$. We claim that R is a residual set. It remains to prove that it is dense in $C(\Omega)$.

Let $p \in \mathcal{P}(\Omega)$. From Corollary 5.2, for each $\omega \in \Omega$ one has $x_1(\omega, \alpha^*(p)) = x_2(\omega, \alpha^*(p))$, which are continuous functions because $M_{\alpha^*(p)}$ is a minimal set. Moreover, $x_1(\omega, \alpha)$ and $x_2(\omega, \alpha)$ are monotone functions on α, as we have checked in Theorem 3.3. Therefore, from Dini's theorem,

$$x_1(\omega, \alpha^*(p)) = \lim_{\alpha \to \alpha^*(p)^-} x_1(\omega, \alpha), \quad x_2(\omega, \alpha^*(p)) = \lim_{\alpha \to \alpha^*(p)^-} x_2(\omega, \alpha),$$

uniformly on Ω. Thus, given n there is $\alpha < \alpha^*(p)$ such that $\|x_2(\cdot, \alpha) - x_1(\cdot, \alpha)\|_\infty < 1/n$ and $p \in V_n$. Finally, from the isomorphism between $C(\Omega)$

and $\mathcal{A}(\mathcal{M}(f))$ it is immediate to check that $\mathcal{P}(\Omega)$ is dense in $C(\Omega)$. Thus, R is dense in $C(\Omega)$, and then a residual set.

In order to finish the proof, we check that we are in situation (c.1) when $p \in R$. We have $p \in V_n$ for each $n \in \mathbb{N}$. Consequently, for each $n \in \mathbb{N}$ there is $\alpha_n < \alpha^*(p)$ such that $\|x_2(\cdot, \alpha_n) - x_1(\cdot, \alpha_n)\|_\infty < 1/n$. In addition, we know that

$$\|x_2(\cdot, \alpha^*(p)) - x_1(\cdot, \alpha^*(p))\|_\infty < \|x_2(\cdot, \alpha_n) - x_1(\cdot, \alpha_n)\|_\infty < 1/n,$$

which implies that $x_2(\omega, \alpha^*(p)) = x_1(\omega, \alpha^*(p))$ for each $\omega \in \Omega$, as claimed. ∎

6. Quasi-periodic case

In this section we are concerned with the quasi-periodic case. We denote by $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$ the standard k-torus with $k \geq 2$. For any vector of frequencies $\phi \in [0,1]^k$, we consider the linear flow $(\mathbb{T}^k, \phi)$ defined by translation on $\mathbb{T}^k$ as $\omega \cdot t = \omega + t\phi \pmod 1$, $\omega \in \mathbb{T}^k$, $t \in \mathbb{R}$. It is well known that if the components of the vector of frequencies are rationally independent, the flow is minimal. In the general case, this is not necessarily true, but we can apply the results given in the previous sections to each of the minimal subsets in which the flow $(\mathbb{T}^k, \phi)$ decomposes.

The next result shows that the hyperbolic solutions inherit the regularity of the equation.

THEOREM 6.1: *Let us assume that $x_0 \in C(\mathbb{T}^k)$ is a hyperbolic solution of*

$$x' = g(x) + p(\omega + t\phi), \quad \omega \in \mathbb{T}^k, \tag{6.1}$$

where $g \in C^1(\mathbb{R})$ and $p \in C^1(\mathbb{T}^k)$. Then, $x_0 \in C^1(\mathbb{T}^k)$.

Proof: We denote by $x(t, \omega, r)$ the solution of equation (6.1) with initial condition $x(0, \omega, r) = r$. Since x_0 is hyperbolic, we can assume that $x_0(\omega + t\phi)$ is uniformly asymptotically stable as t goes to $+\infty$, otherwise it would be as t goes to $-\infty$ and the proof is similar. Therefore, since it can be shown that the modulus of uniform stability is independent of ω, there is a $\delta > 0$ such that whenever $|x_0(\omega) - r| < \delta$, then $\lim_{t\to\infty}[x(t, \omega, r) - x_0(\omega + t\phi)] = 0$.

Moreover, if we fix $\omega_0 \in \mathbb{T}^k$ and r satisfying $|x_0(\omega_0) - r| < \delta$, we can find a neighborhood $B(\omega_0, \rho)$ such that $|x_0(\omega) - r| < \delta$ for every $\omega \in B(\omega_0, \rho)$, and thus

$$\lim_{t\to\infty}[x(t, \omega, r) - x_0(\omega + t\phi)] = 0 \quad \text{uniformly in } B(\omega_0, \rho). \tag{6.2}$$

We can also take a sequence $t_n \uparrow \infty$ such that $\lim_{n\to\infty} t_n\phi = 0$. Consequently, $\omega + t_n\phi \to \omega$ uniformly on $\mathbb{T}^k$, and if we define the functions $f_n(\omega) = x(t_n, \omega, r)$, we deduce that $x_0(\omega) = \lim_{n\to\infty} f_n(\omega)$ uniformly on $B(\omega_0, \rho)$. The classical theorems of dependence with respect to parameters show that $f_n \in C^1(\mathbb{T}^k)$. We claim that the sequence of derivatives $\{\nabla f_n\}_{n\in\mathbb{N}}$ converges uniformly on $B(\omega_0, \rho)$ to a continuous function, which would imply that $x_0 \in C^1(\mathbb{T}^k)$, as stated.

For each $\omega \in B(\omega_0, \rho)$, the function $x_\omega(t) = \partial x/\partial\omega_1(t, \omega, r)$ is the solution of

$$x' = g'(x(t,\omega,r))x + \frac{\partial p}{\partial \omega_1}(\omega + t\phi), \quad t \in \mathbb{R},$$

with initial condition $x_\omega(0) = 0$, which can be written as

$$x_\omega(t) = \int_0^t \exp\left[\int_s^t g'(x(\tau,\omega,r))d\tau\right] \frac{\partial p}{\partial \omega_1}(\omega + s\phi)ds, \quad t \in \mathbb{R}.$$

From the exponential dichotomy of the family $z' = g'(x_0(\omega + t\phi))z$, $\omega \in \mathbb{T}^k$, and the limit (6.2), it is not difficult to check that there exist constants $t_0 \in \mathbb{R}$ and $M > 0$ such that $|x_\omega(t)| \le M$ for all $t \ge t_0$ and $\omega \in B(\omega_0, \rho)$.

Next, for each $\omega \in B(\omega_0, \rho)$ we consider the non-homogeneous equation

$$y' = g'(x_0(\omega + t\phi))y + \frac{\partial p}{\partial \omega_1}(\omega + t\phi), \quad t \in \mathbb{R}. \tag{6.3}$$

It is immediate to check that for each $y(t)$ solution of equation (6.3), the function $z(t) = x_\omega(t) - y(t)$ is a solution of

$$z' = g'(x_0(\omega + t\phi))z + [g'(x(t,\omega,r)) - g'(x_0(\omega + t\phi))]x_\omega(t). \tag{6.4}$$

In particular, for each $t_1 \ge t_0$ we consider the solution of (6.4) given for $t \ge t_1$ by

$$z_\omega(t) = \int_{t_1}^t \exp\left[\int_s^t g'(x_0(\omega + \tau\phi))d\tau\right][g'(x(s,\omega,r)) - g'(x_0(\omega + s\phi))]x_\omega(s)ds,$$

and $y_\omega(t) = x_\omega(t) - z_\omega(t)$. Moreover, from the uniform asymptotic stability of $x_0(\omega + t\phi)$ as t goes to $+\infty$, the linear part $z' = g'(x_0(\omega + t\phi))z$ admits an exponential dichotomy with projection $P = I$. Then, for every $t \ge t_1$

$$|z_\omega(t)| \le K \sup_{t_1 \le s \le t}\{|[g'(x(s,\omega,r)) - g'(x_0(\omega + s\phi))]x_\omega(s)|\},$$

where K is a constant which only depends on the exponential dichotomy.

Besides, from (6.2) and the continuity of g', given $\varepsilon > 0$ there is a $t_1 \geq t_0$ such that $\sup_{t_1 \leq s \leq t}\{|g'(x(s,\omega,r)) - g'(x_0(\omega+s\phi))|\} \leq \varepsilon/(2MK)$, and consequently, $|z_\omega(t)| \leq \varepsilon/2$, that is, $|x_\omega(t) - y_\omega(t)| \leq \varepsilon/2$ for all $\omega \in B(\omega_0,\rho)$ and $t \geq t_1$.

Finally, we can denote by $q(\omega+t\phi)$ the unique bounded solution of(6.3) where $q \in C(\mathbb{T}^k)$. Then, $y_\omega(t) - q(\omega+t\phi)$ is a solution of the linear equation $z' = g'(x_0(\omega+t\phi))z$, whose solutions, when the initial data are uniformly bounded, tend to zero uniformly on $\mathbb{T}^k$ as $t \to \infty$ because of the exponential dichotomy. Thus, there is a $t_2 \geq t_1$ such that $|y_\omega(t) - q(\omega+t\phi)| < \varepsilon/2$ for each $t \geq t_2$ and $\omega \in \mathbb{T}^k$. Hence, $|x_\omega(t) - q(\omega+t\phi)| < \varepsilon$ for each $t \geq t_2$, $\omega \in B(\omega_0,\rho)$, and we conclude that $\partial f_n/\partial\omega_1(\omega) = x_\omega(t_n)$ converges to $q(\omega)$ uniformly on $B(\omega_0,\rho)$. The same can be done for the rest of the partial derivatives, and the proof is finished, as explained above. ∎

The following perturbation result asserts that hyperbolicity is maintained in an appropriate neighborhood of the equation and the frequency. Notice that the function p is assumed to be continuously differentiable on the k-torus.

THEOREM 6.2: *Let us assume that $x_0 \in C(\mathbb{T}^k)$ is a hyperbolic solution of*

$$x' = g(x) + p(\omega+t\phi), \quad \omega \in \mathbb{T}^k,$$

where $g \in C^1(\mathbb{R})$ and $p \in C^1(\mathbb{T}^k)$. Then, for each $\delta > 0$, there is an $\varepsilon(\delta) > 0$ such that if $\phi_1 \in [0,1]^k$, $p_1 \in C(\mathbb{T}^k)$ satisfy $\|\phi - \phi_1\| < \varepsilon(\delta)$ and $\|p - p_1\|_\infty < \varepsilon(\delta)$, the perturbed family

$$x' = g(x) + p_1(\omega+t\phi_1), \quad \omega \in \mathbb{T}^k,$$

admits a hyperbolic solution $x_1 \in C(\mathbb{T}^k)$ with $\|x_0 - x_1\|_\infty < \delta$.

Proof: For each $\omega \in \mathbb{T}^k$ and $t \in \mathbb{R}$ we denote $y_{\phi,\omega}(t) = g'(x_0(\omega+t\phi))$, and we define the bounded and invariant under translations sets of real maps

$$Y_\phi = \{y_{\phi,\omega} \mid \omega \in \mathbb{T}^k\}, \quad Z_{\phi,r} = \{y_{\phi_1,\omega} \mid \omega \in \mathbb{T}^k \text{ and } \|\phi - \phi_1\| \leq r\} \subset L^\infty(\mathbb{R}).$$

Let $r_0 > 0$ be such that Y_ϕ, $Z_{\phi,r} \subset B_{r_0} = \{x \in L^\infty(\mathbb{R}) \mid \|x\|_\infty \leq r_0\}$. We endow $L^\infty(\mathbb{R})$ with the weak* topology $\sigma(L^\infty(\mathbb{R}), L^1(\mathbb{R}))$. Since $L^1(\mathbb{R})$ is separable, $B_{r_0} \subset L^\infty(\mathbb{R})$ is a compact metrizable space. Then, Y_ϕ and $Z_{\phi,r}$ are compact translation invariant subsets. Moreover, because of the hyperbolicity of x_0, the family of equations $x' = y(t)x$, $y \in Y_\phi$ admits an exponential dichotomy over Y_ϕ. Therefore, for r small enough so that the Hausdorff distance between Y_ϕ and $Z_{\phi,r}$ is also small enough, the perturbation theorem of Sacker and Sell [26] ensures the exponential dichotomy over $Z_{\phi,r}$ for the equations $x' = z(t)x$, $z \in Z_{\phi,r}$.

From Theorem 6.1 we know that $x_0 \in C^1(\mathbb{T}^k)$. Then, for $\phi_1 \in [0,1]^k$ with $\|\phi-\phi_1\| < r$ we consider $q_{\phi_1} \in C(\mathbb{T}^k)$ defined by $q_{\phi_1}(\omega) = \nabla x_0(\omega)\cdot\phi_1 - g(x_0(\omega))$. An easy calculation shows that $x_0(\omega + t\phi_1)$ satisfies, for each $\omega \in \mathbb{T}^k$, the equation $x' = g(x) + q_{\phi_1}(\omega + t\phi_1)$. Consequently, it is a hyperbolic solution of this family because $x' = g'(x_0(\omega + t\phi_1))x$, $\omega \in \mathbb{T}^k$, admits an exponential dichotomy, as shown above. Moreover, notice that

$$\|q_{\phi_1} - p\|_\infty \leq \|\nabla x_0\|_\infty \|\phi - \phi_1\|.$$

Next, from Theorem 2.1 we deduce that given $\delta > 0$ there is an $\varepsilon_1(\delta) > 0$ such that, whenever $p_1 \in C(\mathbb{T}^k)$ with $\|p_1 - q_{\phi_1}\|_\infty < \varepsilon_1(\delta)$ and $\|\phi - \phi_1\| < r$, there is a hyperbolic solution $x_1 \in C(\mathbb{T}^k)$ for the equations

$$x' = g(x) + p_1(\omega + t\phi_1), \quad \omega \in \mathbb{T}^k,$$

satisfying $\|x_0 - x_1\|_\infty < \delta$. Notice that $\varepsilon_1(\delta)$ can be chosen independent of ϕ_1 when $\|\phi - \phi_1\| < r$, because the constants of the exponential dichotomy are common over $Z_{\phi,r}$. Finally, we can choose $0 < r_1 \leq r$ and $s_1 > 0$ such that $s_1 + \|\nabla x_0\|_\infty r_1 < \varepsilon_1(\delta)$ and take $\varepsilon(\delta) = \min(r_1, s_1)$ to finish the proof. ∎

Given a real function g defined on the real line satisfying Assumption 3.1 and a real function $p \in C(\mathbb{T}^k)$, we consider the family of differential equations

$$x' = g(x) + p(\omega + t\phi) + \alpha^*(\omega, \phi, p), \quad \omega \in \mathbb{T}^k, \tag{6.5}$$

where $\alpha^*(\omega, \phi, p)$ is the value obtained in Theorem 3.3, which is constant, as a function of ω, over each minimal component of the flow $(\mathbb{T}^k, \phi)$. In general it is not constant over the whole torus, but from Proposition 4.2 it defines a continuous map on ω for any fixed pair $(\phi, p) \in [0,1]^k \times C(\mathbb{T}^k)$.

LEMMA 6.3: *Let $\alpha \in \mathbb{R}$ be such that $\alpha < \alpha^*(\omega, \phi, p)$ for each $\omega \in \mathbb{T}^k$ and some $\phi \in [0,1]^k$ and $p \in C(\mathbb{T}^k)$ fixed. Then*

$$x' = g(x) + p(\omega + t\phi) + \alpha, \quad \omega \in \mathbb{T}^k, \tag{6.6}$$

has two hyperbolic solutions $x_1(\omega, \phi, p, \alpha)$ and $x_2(\omega, \phi, p, \alpha)$.

Proof: The result is immediate from the definition and properties of $\alpha^*(\omega, \phi, p)$ when the components of ϕ are rationally independent. In the rationally dependent case, we know that there are two hyperbolic solutions $x_1(\omega, \phi, p, \alpha)$ and $x_2(\omega, \phi, p, \alpha)$ in each minimal component of the flow $(\mathbb{T}^k, \phi)$. Besides, from the compactness of $\mathbb{T}^k$, it is easy to check that the exponential dichotomy of the corresponding linearized systems holds over $\mathbb{T}^k$. Thus, to finish the proof it suffices to show that $x_1(\omega, \phi, p, \alpha)$ and $x_2(\omega, \phi, p, \alpha)$ are continuous in $\mathbb{T}^k$, as functions of ω.

For each $\omega \in \mathbb{T}^k$ fixed, we consider the function q_ω defined by $q_\omega(t\phi) = p(\omega + t\phi)$ in the torus of smaller dimension $\mathbb{T}^s$. Moreover, given $\varepsilon > 0$ there is a $\delta > 0$ such that if $d(\omega, \omega') < \delta$, then $\|q_\omega - q_{\omega'}\|_\infty < \varepsilon$, and the result follows from an application of the perturbation Theorem 2.1 in $\mathbb{T}^s$. ∎

Now we study more continuity properties for α^*. In particular, the next result holds for any ϕ_0 with rationally independent components.

Proposition 6.4: *Let us assume that $\alpha^*(\omega, \phi_0, p_0)$ is constant in $\mathbb{T}^k$ for some fixed $\phi_0 \in [0,1]^k$ and $p_0 \in C(\mathbb{T}^k)$. Then, the function*

$$\alpha^*\colon \mathbb{T}^k \times [0,1]^k \times C(\mathbb{T}^k) \to \mathbb{R}, \quad (\omega, \phi, p) \mapsto \alpha^*(\omega, \phi, p)$$

is continuous at (ϕ_0, p_0) uniformly on $\mathbb{T}^k$, i.e., for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $\|\phi - \phi_0\| < \delta$ and $\|p - p_0\|_\infty < \delta$, then

$$|\alpha^*(\omega, \phi, p) - \alpha^*(\omega, \phi_0, p_0)| < \varepsilon \quad \text{for each } \omega \in \mathbb{T}^k.$$

Proof: We denote by α_0^* the constant value $\alpha^*(\omega, \phi_0, p_0)$, $\omega \in \mathbb{T}^k$. From Proposition 4.2 we deduce that

$$(6.7)\quad |\alpha^*(\omega, \phi, p) - \alpha^*(\omega, \phi, q)| \le \|p - q\|_\infty \quad \text{if } p, q \in C(\mathbb{T}^k), \omega \in \mathbb{T}^k, \phi \in [0,1]^k.$$

From this fact and the constancy of $\alpha^*(\omega, \phi_0, p_0)$, it is not difficult to show that it suffices to prove that if $\omega_n \to \omega$ and $\phi_n \to \phi_0$, then $\alpha^*(\omega_n, \phi_n, p_0) \to \alpha_0^*$.

From Proposition 4.3 we already know that $\limsup_{n\to\infty} \alpha^*(\omega_n, \phi_n, p_0) \le \alpha_0^*$. Therefore, argue by contradiction and assume that there is a subsequence $\{n_j\}$ such that $\lim_{j\to\infty} \alpha^*(\omega_{n_j}, \phi_{n_j}, p_0) = \alpha_0 < \alpha_0^*$. Choose α, β with $\alpha_0 < \alpha < \beta < \alpha_0^*$ and find j_0 such that if $j \ge j_0$, then $\alpha^*(\omega_{n_j}, \phi_{n_j}, p_0) < \alpha$.

From (6.7), the density of $C^1(\mathbb{T}^k)$ in $C(\mathbb{T}^k)$ and $\beta < \alpha_0^* = \alpha^*(\omega, \phi_0, p_0)$, we can also obtain $p \in C^1(\mathbb{T}^k)$ such that $\|p - p_0\|_\infty < (\beta - \alpha)/2$ and $\beta < \alpha^*(\omega, \phi_0, p)$ for each $\omega \in \mathbb{T}^k$. Therefore, from Lemma 6.3, $x' = g(x) + p(\omega + t\phi_0) + \beta$ has two hyperbolic solutions, and since $p \in C^1(\mathbb{T}^k)$, Theorem 6.2 shows that there is $j_1 \ge j_0$ such that the same happens for $x' = g(x) + p(\omega_{n_j} + t\phi_{n_j}) + \beta$ when $j \ge j_1$.

Thus, $\alpha^*(\omega_{n_j}, \phi_{n_j}, p_0) < \alpha < \beta \le \alpha^*(\omega_{n_j}, \phi_{n_j}, p)$ for each $j \ge j_1$, which contradicts that $|\alpha^*(\omega_{n_j}, \phi_{n_j}, p) - \alpha^*(\omega_{n_j}, \phi_{n_j}, p_0)| \le \|p - p_0\|_\infty < (\beta - \alpha)/2$, and finishes the proof. ∎

PROPOSITION 6.5: *Let $(\phi,p)\in[0,1]^k\times C(\mathbb{T}^k)$ be such that the components of ϕ are rationally dependent and $\alpha^*(\omega,\phi,p)=\alpha^*(\phi,p)$ for every $\omega\in\mathbb{T}^k$, i.e., α^* is constant as a function of ω. Then, given $\varepsilon>0$ there is an $\alpha(\varepsilon)<\alpha^*(\phi,p)$ such that*

$$|x_1(\omega,\phi,p,\alpha(\varepsilon))-x_2(\omega,\phi,p,\alpha(\varepsilon))|<\varepsilon \quad \textit{for each } \omega\in\mathbb{T}^k.$$

Proof: As the components of ϕ are rationally dependent, we can find a value $T>0$ such that $p(\omega+T\phi)=p(\omega)$ for every $\omega\in\mathbb{T}^k$, that is, we can say that p is periodic on the torus for the flow given by ϕ.

Then, as in Section 5, we deduce that

$$x_1(\omega,\phi,p,\alpha^*(\phi,p))=x_2(\omega,\phi,p,\alpha^*(\phi,p))$$

for every $\omega\in\mathbb{T}^k$. Moreover, since α^* is independent of $\omega\in\mathbb{T}^k$, it can be shown that they are continuous functions over the torus. Therefore, as in Theorem 5.3, from Dini's theorem we conclude that

$$\lim_{\alpha\to\alpha^*(\phi,p)^-} x_i(\omega,\phi,p,\alpha)=x_i(\omega,\phi,p,\alpha^*(\phi,p)), \quad i=1,2$$

uniformly in $\mathbb{T}^k$, which shows the result. ∎

Finally, we show that situation (c.1) of Proposition 3.4 is generic in the sense stated in the following result.

THEOREM 6.6: *There exists a residual subset $R\subset[0,1]^k\times C(\mathbb{T}^k)$ such that for any $(\phi,p)\in R$, the function $\alpha^*(\omega,\phi,p)$ is constant in $\mathbb{T}^k$, i.e., $\alpha^*(\omega,\phi,p)=\alpha^*(\phi,p)$ for each $\omega\in\mathbb{T}^k$ and*

$$x_1(\omega,\phi,p,\alpha^*(\phi,p))=x_2(\omega,\phi,p,\alpha^*(\phi,p)) \quad \textit{for each } \omega\in\mathbb{T}^k,$$

where the above functions are defined for equations (6.5) as in Theorem 3.3.

Proof: For each $n\in\mathbb{N}$ we define the sets

$$\begin{aligned} V_n=\{(\phi,p)\in[0,1]^k\times C(\mathbb{T}^k)\ |\exists\alpha\in\mathbb{R} \text{ such that } \alpha<\alpha^*(\omega,\phi,p)\forall\omega\in\mathbb{T}^k,\\ \text{and } \|x_1(\cdot,\phi,p,\alpha)-x_2(\cdot,\phi,p,\alpha)\|_\infty<1/n\}\end{aligned}$$

and $V_n^1=V_n\cap([0,1]^k\times C^1(\mathbb{T}^k))$. Given $r>0$ and $(\phi,p)\in[0,1]^k\times C(\mathbb{T}^k)$ we denote by

$$B((\phi,p),r)=\{(\phi_1,p_1)\in[0,1]^k\times C(\mathbb{T}^k)\ |\ \|\phi-\phi_1\|<r,\|p-p_1\|_\infty<r\}.$$

Next, we claim that for each $(\phi,p) \in V_n^1$ there is an $\varepsilon_n(\phi,p) > 0$ such that

$$B((\phi,p),\varepsilon_n(\phi,p)) \subset V_n.$$

Since $(\phi,p) \in V_n$, from Lemma 6.3 we know that $x_1(\omega,\phi,p,\alpha)$ and $x_2(\omega,\phi,p,\alpha)$ are hyperbolic solutions of (6.6). Moreover, from $p \in C^1(\mathbb{T}^k)$ and Theorem 6.2, there is an $\varepsilon_n(\phi,p) > 0$ such that if $\|\phi-\phi_1\| < \varepsilon_n(\phi,p)$ and $\|p-p_1\|_\infty < \varepsilon_n(\phi,p)$

$$x' = g(x) + p_1(\omega + t\phi_1) + \alpha, \quad \omega \in \mathbb{T}^k$$

admits two hyperbolic solutions $x_1(\omega,\phi_1,p_1,\alpha)$ and $x_2(\omega,\phi_1,p_1,\alpha)$ satisfying

$$\|x_1(\cdot,\phi_1,p_1,\alpha) - x_2(\cdot,\phi_1,p_1,\alpha)\|_\infty < 1/n,$$

that is, $(\phi_1,p_1) \in V_n$ as claimed.

For each $n \in \mathbb{N}$, we define the open subset of $[0,1]^k \times C(\mathbb{T}^k)$

$$W_n = \bigcup_{(\phi,p)\in V_n^1} B((\phi,p),\varepsilon_n(\phi,p)) \subset V_n,$$

and we consider the G_δ set $R = \bigcap_{n\in\mathbb{N}} W_n$. In order to obtain the residual character of the set R, it suffices to prove that each W_n is a dense set. Moreover, since $V_n^1 \subset W_n$, it is enough to show that each V_n^1 is dense in $[0,1]^k \times C(\mathbb{T}^k)$.

Thus, let us fix an $n \in \mathbb{N}$ and a pair $(\phi,p) \in [0,1]^k \times C^1(\mathbb{T}^k)$ such that the components of ϕ are rationally independent. We will show that given $\delta > 0$ there is an element $(\phi_1,p_1) \in V_n^1$ with $\|\phi_1 - \phi\| < \delta$ and $\|p_1 - p\|_\infty < \delta$.

First, we take a sequence of frequencies with rationally dependent components $\{\phi_j\}_{j\in\mathbb{N}} \subset [0,1]^k$ such that $\lim_{j\to\infty} \phi_j = \phi$. By Proposition 6.4, we find a j_0 with $|\alpha^*(\omega,\phi_j,p) - \alpha^*(\phi,p)| < \delta$ for each $j \geq j_0$ and $\omega \in \mathbb{T}^k$.

Next, we fix $j_1 \geq j_0$ such that $\|\phi_{j_1} - \phi\| < \delta$, and consider the function

$$q(\omega) = p(\omega) + \alpha^*(\omega,\phi_{j_1},p) - \alpha^*(\phi,p), \quad \omega \in \mathbb{T}^k,$$

which is continuous, and we can define the corresponding value $\alpha^*(\omega,\phi_{j_1},q)$. Notice that the points ω and $\omega + t\phi_{j_1}$ are in the same orbit of the flow $(\mathbb{T}^k,\phi_{j_1})$, so that $\alpha^*(\omega + t\phi_{j_1},\phi_{j_1},p) = \alpha^*(\omega,\phi_{j_1},p)$. Hence

$$\begin{aligned} x' &= g(x) + q(\omega + t\phi_{j_1}) + \alpha \\ &= g(x) + p(\omega + t\phi_{j_1}) + \alpha^*(\omega,\phi_{j_1},p) - \alpha^*(\phi,p) + \alpha, \quad \omega \in \mathbb{T}^k, \end{aligned}$$

from which we deduce that $\alpha^*(\omega,\phi_{j_1},q) = \alpha^*(\phi,p)$ for every $\omega \in \mathbb{T}^k$. Now Proposition 6.5 shows that $(\phi_{j_1},q) \in V_n$. Thus, from Theorem 2.1, it is easy to check that there is a positive constant $c(\phi_{j_1}, q) > 0$ such that if $p_1 \in C(\mathbb{T}^k)$ and satisfies $\|q - p_1\|_\infty < c(\phi_{j_1}, q)$, then $(\phi_{j_1}, p_1) \in V_n$.

Let $\delta_0 = \|q-p\|_\infty = \|\alpha^*(\cdot, \phi_{j_1}, p) - \alpha^*(\phi, p)\|_\infty < \delta$. There exists $p_1 \in C^1(\mathbb{T}^k)$ such that $\|q - p_1\|_\infty < \min(\delta - \delta_0, c(\phi_{j_1}, q))$. Consequently, $(\phi_{j_1}, p_1) \in V_n^1$ with $\|\phi_{j_1} - \phi\| < \delta$, $\|p_1 - p\|_\infty < \delta$ as stated, and $R = \bigcap_{n\in\mathbb{N}} W_n$ is a residual subset.

Finally, we show that the statements hold for each $(\phi, p) \in R$. From $W_n \subset V_n$ we deduce that there is an α_n such that $\alpha_n \leq \alpha^*(\omega, \phi, p)$ for each $\omega \in \mathbb{T}^k$ and

$$\|x_1(\cdot, \phi, p, \alpha_n) - x_2(\cdot, \phi, p, \alpha_n)\|_\infty < 1/n,$$

from which it is easily deduced that $\alpha^*(\omega, \phi, p)$ is constant in $\mathbb{T}^k$ because it coincides with $\sup_{n\in\mathbb{N}} \alpha_n$, and $x_1(\omega, \phi, p, \alpha^*(\phi, p)) = x_2(\omega, \phi, p, \alpha^*(\phi, p))$ for each $\omega \in \mathbb{T}^k$, which finishes the proof. ∎

References

[1] A. I. Alonso and R. Obaya, *The structure of the bounded trajectories set of a scalar convex differential equation,* Proceedings of the Royal Society of Edinburgh. Section A **133** (2003), 237–263.

[2] A. I. Alonso, R. Obaya and R. Ortega, *Differential equations with limit-periodic forcings,* Proceedings of the American Mathematical Society **131** (2002), 851–857.

[3] A. Berger, S. Siegmund and Y. Yi, *On almost automorphic dynamics in symbolic lattices,* Ergodic Theory and Dynamical Systems **24** (2004), 677–696.

[4] S. Bochner, *Curvature and Betti number in real and complex vector bundles,* Rendiconti del Seminario Mathematico della Universitá e Politecnico di Torino **15** (1956), 225–253.

[5] W. A. Coppel, *Dichotomies in Stability Theory,* Lecture Notes in Mathematics **629**, Springer-Verlag, Berlin, Heildelberg, New York, 1978.

[6] R. Ellis, *Lectures on Topological Dynamics,* Benjamin, New York, 1969.

[7] A. M. Fink, *Almost Periodic Differential Equations,* Lecture Notes in Mathematics **377**, Springer-Verlag, Berlin, Heildelberg, New York, 1974.

[8] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation,* Mathematical Systems Theory **1** (1967), 1–49.

[9] H. Furstenberg and B. Weiss, *On almost 1-1 extensions,* Israel Journal of Mathematics **65** (1989), 311–322.

[10] R. Johnson, *On a Floquet theory for almost-periodic, two-dimensional linear systems,* Journal of Differential Equations **37** (1980), 184–205.

[11] R. Johnson, *A linear, almost periodic equation with an almost automorphic solution,* Proceedings of the American Mathematical Society **82** (1981), 199–205.

[12] R. Johnson and J. Moser, *The rotation number for almost periodic differential equations,* Communications in Mathematical Physics **84** (1982), 403–438.

[13] R. Johnson, S. Novo and R. Obaya, *An ergodic and topological approach to disconjugate linear hamiltonian systems,* Illinois Journal of Mathematics **45** (2001), 803–822.

[14] R. Johnson, S. Novo and R. Obaya, *Ergodic properties and Weyl M-functions for random linear Hamiltonian systems,* Proceedings of the Royal Society of Edinburgh. Section A **130** (2000), 1045–1079.

[15] R. Mañé, *Ergodic Theory and Differentiable Dynamics,* Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[16] N. G. Markley and M. E. Paul, *Almost automorphic symbolic minimal sets without unique ergodicity,* Israel Journal of Mathematics **34** (1979), 259–272.

[17] J. L. Massera, *The existence of periodic solutions of systems of differential equations,* Duke Mathematical Journal **17** (1950), 457–475.

[18] J. Mawhin, *First order ordinary differential equations with several periodic solutions,* Zeitschrift für angewandte Mathematik und Physik **38** (1987), 257–265.

[19] V. M. Millionščikov, *Proof of the existence of irregular systems of linear differential equations with almost periodic coefficients,* Differential Equations **4** (1968), 391–396.

[20] V. M. Millionščikov, *Proof of the existence of irregular systems of linear differential equations with quasi periodic coefficients,* Differential Equations **5** (1969), 1475–1478.

[21] V. Nemytskii and V. Stepanoff, *Qualitative Theory of Differential Equations,* Princeton University Press, Princeton, NJ, 1960.

[22] S. Novo and R. Obaya, *Strictly ordered minimal subsets of a class of convex monotone skew-product semiflows,* Journal of Differential Equations **196** (2004), 249–288.

[23] R. Ortega and M. Tarallo, *Almost periodic linear differential equations with non-separated solutions,* `http://www.ugr.es/~ecuadif/fuentenueva.htm`.

[24] R. Phelps, *Lectures on Choquet's Theory,* Van Nostrand Mathematical Studies, American Book Co., New York, 1966.

[25] R. J. Sacker and G. R. Sell, *Lifting properties in skew-products flows with applications to differential equations,* Memoirs of the American Mathematical Society **190**, American Mathematical Society, Providence, RI, 1977.

[26] R. J. Sacker and G. R. Sell, *A spectral theory for linear differential systems,* Journal of Differential Equations **27** (1978), 320–358.

[27] W. Shen and Y. Yi, *Dynamics of almost periodic scalar parabolic equations,* Journal of Differential Equations **122** (1995), 114–136.

[28] W. Shen and Y. Yi, *Almost Automorphic and Almost Periodic Dynamics in Skew-Products Semiflows,* Memoirs of the American Mathematical Society **647**, American Mathematical Society, Providence, RI, 1998.

[29] A. Tineo, *First-order ordinary differential equations with several bounded separate solutions,* Journal of Mathematical Analysis and Applications **225** (1998), 359–372.

[30] W. A. Veech, *Almost automorphic functions on groups,* American Journal of Mathematics **87** (1965), 719–751.

[31] W. A. Veech, *Topological dynamics,* Bulletin of the American Mathematical Society **83** (1977), 775–830.

[32] R. E. Vinograd, *A problem suggested by N. P. Erugin,* Differentsial'nye Uravneniya **11** (1975), 632–638.

[33] V. V. Zhikov and B. M. Levitan, *Favard theory,* Russian Mathematical Surveys **32** (1977), 129–180.